

CHAPTER 67 INTRODUCTION TO LAPLACE TRANSFORMS

EXERCISE 247 Page 729

1. Determine the Laplace transforms of: (a) $2t - 3$ (b) $5t^2 + 4t - 3$

$$(a) \mathcal{L}\{2t - 3\} = 2\left(\frac{1}{s^2}\right) - 3\left(\frac{1}{s}\right) = \frac{2}{s^2} - \frac{3}{s}$$

$$(b) \mathcal{L}\{5t^2 + 4t - 3\} = 5\left(\frac{2!}{s^3}\right) + 4\left(\frac{1}{s^2}\right) - 3\left(\frac{1}{s}\right) = \frac{10}{s^3} + \frac{4}{s^2} - \frac{3}{s}$$

2. Determine the Laplace transforms of: (a) $\frac{t^3}{24} - 3t + 2$ (b) $\frac{t^5}{15} - 2t^4 + \frac{t^2}{2}$

$$(a) \mathcal{L}\left\{\frac{t^3}{24} - 3t + 2\right\} = \left(\frac{1}{24}\right)\left(\frac{3!}{s^{3+1}}\right) - 3\left(\frac{1}{s^2}\right) + 2\left(\frac{1}{s}\right) = \frac{1}{4s^4} - \frac{3}{s^2} + \frac{2}{s}$$

$$(b) \mathcal{L}\left\{\frac{t^5}{15} - 2t^4 + \frac{t^2}{2}\right\} = \left(\frac{1}{15}\right)\left(\frac{5!}{s^{5+1}}\right) - 2\left(\frac{4!}{s^{4+1}}\right) + \left(\frac{1}{2}\right)\left(\frac{2!}{s^3}\right) = \frac{8}{s^6} - \frac{48}{s^5} + \frac{1}{s^3}$$

3. Determine the Laplace transforms of: (a) $5e^{3t}$ (b) $2e^{-2t}$

$$(a) \mathcal{L}\{5e^{3t}\} = 5\left(\frac{1}{s-3}\right) = \frac{5}{s-3}$$

$$(b) \mathcal{L}\{2e^{-2t}\} = 2\left(\frac{1}{s+2}\right) = \frac{2}{s+2}$$

4. Determine the Laplace transforms of: (a) $4 \sin 3t$ (b) $3 \cos 2t$

$$(a) \mathcal{L}\{4 \sin 3t\} = 4\left(\frac{3}{s^2 + 3^2}\right) = \frac{12}{s^2 + 9}$$

$$(b) \mathcal{L}\{3 \cos 2t\} = 3\left(\frac{s}{s^2 + 2^2}\right) = \frac{3s}{s^2 + 4}$$

5. Determine the Laplace transforms of: (a) $7 \cosh 2x$ (b) $\frac{1}{3} \sinh 3t$

$$(a) \mathcal{L}\{7 \cosh 2x\} = 7 \left(\frac{s}{s^2 - 2^2} \right) = \frac{7s}{s^2 - 4}$$

$$(b) \mathcal{L}\left\{\frac{1}{3} \sinh 3t\right\} = \frac{1}{3} \left(\frac{3}{s^2 - 3^2} \right) = \frac{1}{s^2 - 9}$$

6. Determine the Laplace transforms of: (a) $2 \cos^2 t$ (b) $3 \sin^2 2t$

$$(a) \cos 2t = 2 \cos^2 t - 1 \quad \text{from which,} \quad \cos^2 t = \frac{1 + \cos 2t}{2}$$

$$\begin{aligned} \mathcal{L}\{2 \cos^2 t\} &= \mathcal{L}\left\{2 \left(\frac{1 + \cos 2t}{2} \right)\right\} = \mathcal{L}\{1 + \cos 2t\} = \frac{1}{s} + \frac{s}{s^2 + 2^2} = \frac{1}{s} + \frac{s}{s^2 + 4} = \frac{(s^2 + 4) + s^2}{s(s^2 + 4)} \\ &= \frac{2s^2 + 4}{s(s^2 + 4)} = \frac{2(s^2 + 2)}{s(s^2 + 4)} \end{aligned}$$

$$(b) \cos 4t = 1 - 2 \sin^2 2t \quad \text{from which,} \quad \sin^2 2t = \frac{1 - \cos 4t}{2}$$

$$\begin{aligned} \mathcal{L}\{3 \sin^2 2t\} &= \mathcal{L}\left\{3 \left(\frac{1 - \cos 4t}{2} \right)\right\} \\ &= \mathcal{L}\left\{\frac{3}{2} - \frac{3}{2} \cos 4t\right\} = \frac{3/2}{s} - \frac{(3/2)s}{s^2 + 4^2} = \frac{3/2}{s} - \frac{(3/2)s}{s^2 + 16} = \frac{\frac{3}{2}(s^2 + 16) - \frac{3}{2}s^2}{s(s^2 + 16)} = \frac{24}{s(s^2 + 16)} \end{aligned}$$

7. Determine the Laplace transforms of: (a) $\cosh^2 t$ (b) $2 \sinh^2 2\theta$

$$(a) \cosh 2t = 2 \cosh^2 t - 1 \quad \text{from which,} \quad \cosh^2 t = \frac{1 + \cosh 2t}{2} = \frac{1}{2} + \frac{1}{2} \cosh 2t$$

$$\mathcal{L}\{\cosh^2 t\} = \mathcal{L}\left\{\frac{1}{2} + \frac{1}{2} \cosh 2t\right\} = \frac{1/2}{s} + \frac{(1/2)s}{s^2 - 2^2} = \frac{\frac{1}{2}(s^2 - 4) + \frac{1}{2}s^2}{s(s^2 - 4)} = \frac{s^2 - 2}{s(s^2 - 4)}$$

$$(b) \cosh 4\theta = 1 + 2 \sinh^2 2\theta \quad \text{from which,} \quad 2 \sinh^2 2\theta = \cosh 4\theta - 1$$

$$\mathcal{L}\{2\sinh^2 2\theta\} = \mathcal{L}\{\cosh 4\theta - 1\} = \frac{s}{s^2 - 4^2} - \frac{1}{s} = \frac{s^2 - (s^2 - 16)}{s(s^2 - 16)} = \frac{16}{s(s^2 - 16)}$$

8. Determine the Laplace transforms of : $4 \sin(at + b)$, where a and b are constants

$$\begin{aligned}\mathcal{L}\{4 \sin(at + b)\} &= \mathcal{L}\{ \sin at \cos b + \cos at \sin b \} = \mathcal{L}\{ (4 \cos b) \sin at + (4 \sin b) \cos at \} \\ &= (4 \cos b) \left(\frac{a}{s^2 + a^2} \right) + (4 \sin b) \left(\frac{s}{s^2 + a^2} \right) \\ &= \frac{4}{s^2 + a^2} (a \cos b + s \sin b)\end{aligned}$$

9. Determine the Laplace transforms of : $3 \cos(\omega t - \alpha)$, where ω and α are constants

$$\begin{aligned}\mathcal{L}\{3 \cos(\omega t - \alpha)\} &= \mathcal{L}\{ \cos \omega t \cos \alpha + \sin \omega t \sin \alpha \} = \mathcal{L}\{ (3 \cos \alpha) \cos \omega t + (3 \sin \alpha) \sin \omega t \} \\ &= (3 \cos \alpha) \left(\frac{s}{s^2 + \omega^2} \right) + (3 \sin \alpha) \left(\frac{\omega}{s^2 + \omega^2} \right) \\ &= \frac{3}{s^2 + \omega^2} (s \cos \alpha + \omega \sin \alpha)\end{aligned}$$

10. Show that $\{\cos^2 3t - \sin^2 3t\} = \frac{s}{s^2 + 36}$

$$\cos 6t = 2\cos^2 3t - 1 \quad \text{from which,} \quad \cos^2 3t = \frac{1 + \cos 6t}{2}$$

$$\text{and } \cos 6t = 1 - 2\sin^2 3t \quad \text{from which,} \quad \sin^2 3t = \frac{1 - \cos 6t}{2}$$

$$\begin{aligned}\text{Hence, } \mathcal{L}(\cos^2 3t - \sin^2 3t) &= \mathcal{L}\left\{\left(\frac{1 + \cos 6t}{2}\right) - \left(\frac{1 - \cos 6t}{2}\right)\right\} = \mathcal{L}\left\{\frac{1}{2} + \frac{\cos 6t}{2} - \frac{1}{2} + \frac{\cos 6t}{2}\right\} \\ &= \mathcal{L}\{\cos 6t\} = \frac{s}{s^2 + 36}\end{aligned}$$

CHAPTER 68 PROPERTIES OF LAPLACE TRANSFORMS

EXERCISE 248 Page 733

1. Determine the Laplace transforms: (a) $2te^{2t}$ (b) t^2e^t

$$(a) \mathcal{L}\{2te^{2t}\} = 2\mathcal{L}\{t^1e^{2t}\} = (2)\frac{1!}{(s-2)^{1+1}} = \frac{2}{(s-2)^2}$$

$$(b) \mathcal{L}\{t^2e^t\} = \frac{2!}{(s-1)^{2+1}} = \frac{2}{(s-1)^3}$$

2. Determine the Laplace transforms: (a) $4t^3e^{-2t}$ (b) $\frac{1}{2}t^4e^{-3t}$

$$(a) \mathcal{L}\{4t^3e^{-2t}\} = 4\left(\frac{3!}{(s+2)^{3+1}}\right) = \frac{24}{(s+2)^4}$$

$$(b) \mathcal{L}\left\{\frac{1}{2}t^4e^{-3t}\right\} = \frac{1}{2}\left(\frac{4!}{(s+3)^{4+1}}\right) = \frac{12}{(s+3)^5}$$

3. Determine the Laplace transforms: (a) $e^t \cos t$ (b) $3e^{2t} \sin 2t$

$$(a) \mathcal{L}\{e^t \cos t\} = \left(\frac{s-1}{(s-1)^2 + 1^2}\right) = \frac{s-1}{s^2 - 2s + 1 + 1} = \frac{s-1}{s^2 - 2s + 2}$$

$$(b) \mathcal{L}\{3e^{2t} \sin 2t\} = 3\left(\frac{2}{(s-2)^2 + 2^2}\right) = \frac{6}{s^2 - 4s + 4 + 4} = \frac{6}{s^2 - 4s + 8}$$

4. Determine the Laplace transforms: (a) $5e^{-2t} \cos 3t$ (b) $4e^{-5t} \sin t$

$$(a) \mathcal{L}\{5e^{-2t} \cos 3t\} = 5\left(\frac{s+2}{(s+2)^2 + 3^2}\right) = \frac{5(s+2)}{s^2 + 4s + 4 + 9} = \frac{5(s+2)}{s^2 + 4s + 13}$$

$$(b) \mathcal{L}\{4e^{-5t} \sin t\} = 4\left(\frac{1}{(s+5)^2 + 1^2}\right) = \frac{4}{s^2 + 10s + 25 + 1} = \frac{4}{s^2 + 10s + 26}$$

5. Determine the Laplace transforms: (a) $2e^t \sin^2 t$ (b) $\frac{1}{2} e^{3t} \cos^2 t$

$$\begin{aligned} \text{(a) } \mathcal{L}\{2e^t \sin^2 t\} &= \mathcal{L}\left\{2e^t \left(\frac{1 - \cos 2t}{2}\right)\right\} = \mathcal{L}\{e^t\} - \mathcal{L}\{e^t \cos 2t\} = \frac{1}{s-1} - \frac{s-1}{(s-1)^2 + 2^2} \\ &= \frac{1}{s-1} - \frac{s-1}{s^2 - 2s + 5} \end{aligned}$$

$$\begin{aligned} \text{(b) } \mathcal{L}\left\{\frac{1}{2} e^{3t} \cos^2 t\right\} &= \mathcal{L}\left\{\frac{1}{2} e^{3t} \left(\frac{1 + \cos 2t}{2}\right)\right\} \\ &= \mathcal{L}\left\{\frac{1}{4} e^{3t}\right\} + \mathcal{L}\left\{\frac{1}{4} e^{3t} \cos 2t\right\} = \frac{1}{4} \left[\frac{1}{s-3} + \frac{s-3}{(s-3)^2 + 2^2} \right] \\ &= \frac{1}{4} \left(\frac{1}{s-3} + \frac{s-3}{s^2 - 6s + 9 + 4} \right) = \frac{1}{4} \left(\frac{1}{s-3} + \frac{s-3}{s^2 - 6s + 13} \right) \end{aligned}$$

6. Determine the Laplace transforms: (a) $e^t \sinh t$ (b) $3e^{2t} \cosh 4t$

$$\text{(a) } \mathcal{L}\{e^t \sinh t\} = \left(\frac{1}{(s-1)^2 - 1^2} \right) = \frac{1}{s^2 - 2s + 1 - 1} = \frac{1}{s^2 - 2s} = \frac{1}{s(s-2)}$$

$$\text{(b) } \mathcal{L}\{3e^{2t} \cosh 4t\} = 3 \left(\frac{s-2}{(s-2)^2 - 4^2} \right) = \frac{3(s-2)}{s^2 - 4s + 4 - 16} = \frac{3(s-2)}{s^2 - 4s - 12}$$

7. Determine the Laplace transforms: (a) $2e^{-t} \sinh 3t$ (b) $\frac{1}{4} e^{-3t} \cosh 2t$

$$\text{(a) } \mathcal{L}\{2e^{-t} \sinh 3t\} = 2 \left(\frac{3}{(s+1)^2 - 3^2} \right) = \frac{6}{s^2 + 2s + 1 - 9} = \frac{6}{s^2 + 2s - 8}$$

$$\text{(b) } \mathcal{L}\left\{\frac{1}{4} e^{-3t} \cosh 2t\right\} = \frac{1}{4} \left(\frac{s+3}{(s+3)^2 - 2^2} \right) = \frac{1}{4} \left(\frac{s+3}{s^2 + 6s + 9 - 4} \right) = \frac{s+3}{4(s^2 + 6s + 5)}$$

8. Determine the Laplace transforms: (a) $2e^t (\cos 3t - 3 \sin 3t)$ (b) $3e^{-2t} (\sinh 2t - 2 \cosh 2t)$

$$\begin{aligned}
 \text{(a) } \mathcal{L}\{2e^t(\cos 3t - 3\sin 3t)\} &= \mathcal{L}\{2e^t \cos 3t\} - \mathcal{L}\{6e^t \sin 3t\} = 2\left(\frac{s-1}{(s-1)^2 + 3^2}\right) - 6\left(\frac{3}{(s-1)^2 + 3^2}\right) \\
 &= \frac{2(s-1)}{s^2 - 2s + 10} - \frac{18}{s^2 - 2s + 10} = \frac{2s - 2 - 18}{s^2 - 2s + 10} = \frac{2s - 20}{s^2 - 2s + 10} \\
 &= \frac{2(s-10)}{s^2 - 2s + 10}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } \mathcal{L}\{3e^{-2t}(\sinh 2t - 2\cosh 2t)\} &= \mathcal{L}\{3e^{-2t} \sinh 2t\} - \mathcal{L}\{6e^{-2t} \cosh 2t\} \\
 &= 3\left(\frac{2}{(s+2)^2 - 2^2}\right) - 6\left(\frac{s+2}{(s+2)^2 - 2^2}\right) \\
 &= \frac{6}{s^2 + 4s + 4 - 4} - \frac{6(s+2)}{s^2 + 4s + 4 - 4} = \frac{6}{s^2 + 4s} - \frac{6(s+2)}{s^2 + 4s} \\
 &= \frac{6 - 6s - 12}{s(s+4)} = \frac{-6s - 6}{s(s+4)} = \frac{-6(s+1)}{s(s+4)}
 \end{aligned}$$

EXERCISE 249 Page 735

1. Derive the Laplace transform of the first derivative from the definition of a Laplace transform.

Hence derive the transform $\mathcal{L}\{1\} = \frac{1}{s}$

Let $f(t) = 1$ then $f'(t) = 0$ and $f(0) = 1$

From equation (3), page 734 of textbook, $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

Hence,
$$\mathcal{L}\{0\} = s\mathcal{L}\{1\} - 1$$

i.e.
$$0 = s\mathcal{L}\{1\} - 1$$

and
$$\mathcal{L}\{1\} = \frac{1}{s}$$

2. Use the Laplace transform of the first derivative to derive the transforms:

(a) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ (b) $\mathcal{L}\{3t^2\} = \frac{6}{s^3}$

(a) Let $f(t) = e^{at}$ then $f'(t) = ae^{at}$ and $f(0) = 1$

From equation (3), page 734 of textbook, $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

Hence,
$$\mathcal{L}\{ae^{at}\} = s\mathcal{L}\{e^{at}\} - 1$$

i.e.
$$1 = (s-a)\mathcal{L}\{e^{at}\}$$

and
$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

(b) Let $f(t) = 3t^2$ then $f'(t) = 6t$ and $f(0) = 0$

Since
$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

then,
$$\mathcal{L}\{6t\} = s\mathcal{L}\{3t^2\} + 0$$

i.e.
$$\frac{6}{s^2} = s\mathcal{L}\{3t^2\}$$

and
$$\mathcal{L}\{3t^2\} = \frac{6}{s^3}$$

3. Derive the Laplace transform of the second derivative from the definition of a Laplace

transform. Hence derive the transform $\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$

Let $f(t) = \sin at$, then $f'(t) = a \cos at$ and $f''(t) = -a^2 \sin at$, $f(0) = 0$ and $f'(0) = a$

From equation (4), page 734 of textbook, $\mathcal{L}\{f''(t)\} = s^2 \mathcal{L}\{f(t)\} - sf(0) - f'(0)$

Hence, $\mathcal{L}\{-a^2 \sin at\} = s^2 \mathcal{L}\{\sin at\} - s(0) - a$

i.e. $-a^2 \mathcal{L}\{\sin at\} = s^2 \mathcal{L}\{\sin at\} - a$

Hence, $a = (s^2 + a^2) \mathcal{L}\{\sin at\}$

and $\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$

4. Use the Laplace transform of the second derivative to derive the transforms:

(a) $\mathcal{L}\{\sinh at\} = \frac{a}{s^2 - a^2}$ (b) $\mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}$

(a) Let $f(t) = \sinh at$ then $f'(t) = a \cosh at$ and $f''(t) = a^2 \sinh at$, $f(0) = 0$ and $f'(0) = a$

$$\mathcal{L}\{f''(t)\} = s^2 \mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

Hence, $\mathcal{L}\{a^2 \sinh at\} = s^2 \mathcal{L}\{\sinh at\} - s(0) - a$

i.e. $a^2 \mathcal{L}\{\sinh at\} = s^2 \mathcal{L}\{\sinh at\} - a$

i.e. $a = (s^2 - a^2) \mathcal{L}\{\sinh at\}$

and $\mathcal{L}\{\sinh at\} = \frac{a}{s^2 - a^2}$

(b) Let $f(t) = \cosh at$ then $f'(t) = a \sinh at$ and $f''(t) = a^2 \cosh at$, $f(0) = 1$ and $f'(0) = 0$

$$\mathcal{L}\{f''(t)\} = s^2 \mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

Hence, $\mathcal{L}\{a^2 \cosh at\} = s^2 \mathcal{L}\{\cosh at\} - s(1) - 0$

i.e. $a^2 \mathcal{L}\{\cosh at\} = s^2 \mathcal{L}\{\cosh at\} - s$

i.e. $s = (s^2 - a^2) \mathcal{L}\{\cosh at\}$

and $\mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}$

EXERCISE 250 Page 736

1. State the initial value theorem. Verify the theorem for the functions

(a) $3 - 4 \sin t$ (b) $(t - 4)^2$ and state their initial values.

The initial value theorem states: $\lim_{t \rightarrow 0} [f(t)] = \lim_{s \rightarrow \infty} [s \mathcal{L}\{f(t)\}]$

(a) Let $f(t) = 3 - 4 \sin t$ then $\mathcal{L}\{f(t)\} = \mathcal{L}\{3 - 4 \sin t\} = \frac{3}{s} - \frac{4}{s^2 + 1}$

$$\begin{aligned} \text{Hence,} \quad \lim_{t \rightarrow 0} [3 - 4 \sin t] &= \lim_{s \rightarrow \infty} \left[s \left(\frac{3}{s} - \frac{4}{s^2 + 1} \right) \right] \\ &= \lim_{s \rightarrow \infty} \left[3 - \frac{4s}{s^2 + 1} \right] \end{aligned}$$

$$\text{i.e.} \quad 3 - 4 \sin 0 = 3 - \frac{\infty}{\infty^2 + 1}$$

$$\text{i.e.} \quad 3 = 3 \quad \text{which verifies the theorem.}$$

The initial value is 3

(b) Let $f(t) = (t - 4)^2 = t^2 - 8t + 16$

$$\text{then} \quad \mathcal{L}\{t^2 - 8t + 16\} = \frac{2}{s^3} - \frac{8}{s^2} + \frac{16}{s}$$

$$\begin{aligned} \text{Hence,} \quad \lim_{t \rightarrow 0} [t^2 - 8t + 16] &= \lim_{s \rightarrow \infty} \left[s \left(\frac{2}{s^3} - \frac{8}{s^2} + \frac{16}{s} \right) \right] \\ &= \lim_{s \rightarrow \infty} \left[\frac{2}{s^2} - \frac{8}{s} + 16 \right] \end{aligned}$$

$$\text{i.e.} \quad 16 = 16 \quad \text{which verifies the theorem}$$

The initial value is 16

2. Verify the initial value theorem for the voltage functions:

(a) $4 + 2 \cos t$ (b) $t - \cos 3t$ and state their initial values.

The initial value theorem states: $\lim_{t \rightarrow 0} [f(t)] = \lim_{s \rightarrow \infty} [s \mathcal{L}\{f(t)\}]$

(a) Let $f(t) = 4 + 2 \cos t$ then $\mathcal{L}\{f(t)\} = \mathcal{L}\{4 + 2 \cos t\} = \frac{4}{s} + \frac{2s}{s^2 + 1}$

Hence,
$$\lim_{t \rightarrow 0} [4 + 2 \cos t] = \lim_{s \rightarrow \infty} \left[s \left(\frac{4}{s} + \frac{2s}{s^2 + 1} \right) \right]$$

$$= \lim_{s \rightarrow \infty} \left[4 + \frac{2s^2}{s^2 + 1} \right]$$

i.e.
$$4 + 2 \cos 0 = 4 + \frac{2\infty^2}{\infty^2 + 1}$$

i.e.
$$4 + 2 = 4 + 2$$

i.e.
$$6 = 6 \quad \text{which verifies the theorem.}$$

The initial value is 6

(b) Let $f(t) = t - \cos 3t$

then
$$\mathcal{L}\{t - \cos 3t\} = \frac{1}{s^2} - \frac{s}{s^2 + 3^2}$$

Hence,
$$\lim_{t \rightarrow 0} [t - \cos 3t] = \lim_{s \rightarrow \infty} \left[s \left(\frac{1}{s^2} - \frac{s}{s^2 + 9} \right) \right]$$

$$= \lim_{s \rightarrow \infty} \left[\frac{1}{s} - \frac{s^2}{s^2 + 9} \right]$$

i.e.
$$0 - \cos 0 = \frac{1}{\infty} - \frac{\infty^2}{\infty^2 + 9}$$

i.e.
$$-1 = -1 \quad \text{which verifies the theorem}$$

The initial value is -1

3. State the final value theorem and state a practical application where it is of use. Verify the theorem for the function $4 + e^{-2t}(\sin t + \cos t)$ representing a displacement and state its final value.

The final value theorem states: $\lim_{t \rightarrow \infty} [f(t)] = \lim_{s \rightarrow 0} [s \mathcal{L}\{f(t)\}]$

The final value theorem is used in investigating the stability of systems such as in automatic aircraft-landing systems.

$$\text{Let } f(t) = 4 + e^{-2t} (\sin t + \cos t) = 4 + e^{-2t} \sin t + e^{-2t} \cos t$$

$$\text{then } \mathcal{L}\{f(t)\} = \frac{4}{s} + \frac{1}{(s+2)^2 + 1} + \frac{s+2}{(s+2)^2 + 1}$$

$$\text{Hence, } \lim_{t \rightarrow \infty} [4 + e^{-2t} \sin t + e^{-2t} \cos t] = \lim_{s \rightarrow 0} \left[s \left(\frac{4}{s} + \frac{1}{(s+2)^2 + 1} + \frac{s+2}{(s+2)^2 + 1} \right) \right]$$

$$= \lim_{s \rightarrow 0} \left[4 + \frac{s}{(s+2)^2 + 1} + \frac{s(s+2)}{(s+2)^2 + 1} \right]$$

$$\text{i.e. } 4 + 0 + 0 = 4 + 0 + 0$$

$$\text{i.e. } 4 = 4 \quad \text{which verifies the theorem}$$

The final value is 4

4. Verify the final value theorem for the function $3t^2 e^{-4t}$ and determine its steady state value.

The final value theorem states: $\lim_{t \rightarrow \infty} [f(t)] = \lim_{s \rightarrow 0} [s \mathcal{L}\{f(t)\}]$

$$\text{Let } f(t) = 3t^2 e^{-4t}$$

$$\text{then } \mathcal{L}\{f(t)\} = 3 \left(\frac{2!}{(s+4)^{2+1}} \right) = \frac{6}{(s+4)^3}$$

$$\text{Hence, } \lim_{t \rightarrow \infty} [3t^2 e^{-4t}] = \lim_{s \rightarrow 0} \left[s \left(\frac{6}{(s+4)^3} \right) \right]$$

$$\text{i.e. } 3t^\infty e^{-4\infty} = 0$$

$$\text{i.e. } 0 = 0 \quad \text{which verifies the theorem}$$

The final value is 0

CHAPTER 69 INVERSE LAPLACE TRANSFORMS

EXERCISE 251 Page 740

1. Determine the inverse Laplace transforms of : (a) $\frac{7}{s}$ (b) $\frac{2}{s-5}$

$$(a) \mathcal{L}^{-1}\left\{\frac{7}{s}\right\} = 7\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 7(1) = 7$$

$$(b) \mathcal{L}^{-1}\left\{\frac{2}{s-5}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} = 2e^{5t}$$

2. Determine the inverse Laplace transforms of : (a) $\frac{3}{2s+1}$ (b) $\frac{2s}{s^2+4}$

$$(a) \mathcal{L}^{-1}\left\{\frac{3}{2s+1}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{2\left(s+\frac{1}{2}\right)}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)}\right\} = \frac{3}{2}e^{-\frac{1}{2}t}$$

$$(b) \mathcal{L}^{-1}\left\{\frac{2s}{s^2+4}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} = 2 \cos 2t$$

3. Determine the inverse Laplace transforms of : (a) $\frac{1}{s^2+25}$ (b) $\frac{4}{s^2+9}$

$$(a) \mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{5}{s^2+5^2}\right\} = \frac{1}{5}\sin 5t$$

$$(b) \mathcal{L}^{-1}\left\{\frac{4}{s^2+9}\right\} = \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} = \frac{4}{3}\sin 3t$$

4. Determine the inverse Laplace transforms of : (a) $\frac{5s}{2s^2+18}$ (b) $\frac{6}{s^2}$

$$(a) \mathcal{L}^{-1}\left\{\frac{5s}{2s^2+18}\right\} = 5\mathcal{L}^{-1}\left\{\frac{s}{2(s^2+9)}\right\} = \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} = \frac{5}{2}\cos 3t$$

$$(b) \mathcal{L}^{-1}\left\{\frac{6}{s^2}\right\} = 6 \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = 6t$$

5. Determine the inverse Laplace transforms of : (a) $\frac{5}{s^3}$ (b) $\frac{8}{s^4}$

$$(a) \mathcal{L}^{-1}\left\{\frac{5}{s^3}\right\} = \frac{5}{2!} \mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} = \frac{5}{2} t^2$$

$$(b) \mathcal{L}^{-1}\left\{\frac{8}{s^4}\right\} = \frac{8}{3!} \mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{4}{3} t^3$$

6. Determine the inverse Laplace transforms of : (a) $\frac{3s}{\frac{1}{2}s^2 - 8}$ (b) $\frac{7}{s^2 - 16}$

$$(a) \mathcal{L}^{-1}\left\{\frac{3s}{\frac{1}{2}s^2 - 8}\right\} = 3 \mathcal{L}^{-1}\left\{\frac{s}{\frac{1}{2}(s^2 - 16)}\right\} = 6 \mathcal{L}^{-1}\left\{\frac{s}{s^2 - 4^2}\right\} = 6 \cosh 4t$$

$$(b) \mathcal{L}^{-1}\left\{\frac{7}{s^2 - 16}\right\} = 7 \mathcal{L}^{-1}\left\{\frac{1}{(s^2 - 4^2)}\right\} = \frac{7}{4} \mathcal{L}^{-1}\left\{\frac{4}{s^2 - 4^2}\right\} = \frac{7}{4} \sinh 4t$$

7. Determine the inverse Laplace transforms of : (a) $\frac{15}{3s^2 - 27}$ (b) $\frac{4}{(s-1)^3}$

$$(a) \mathcal{L}^{-1}\left\{\frac{15}{3s^2 - 27}\right\} = 15 \mathcal{L}^{-1}\left\{\frac{1}{3\left(s^2 - \frac{27}{3}\right)}\right\} = 5 \mathcal{L}^{-1}\left\{\frac{1}{s^2 - 3^2}\right\} = \frac{5}{3} \mathcal{L}^{-1}\left\{\frac{3}{s^2 - 3^2}\right\} = \frac{5}{3} \sinh 3t$$

$$(b) \mathcal{L}^{-1}\left\{\frac{4}{(s-1)^3}\right\} = 4 \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^{2+1}}\right\} = \frac{4}{2!} \mathcal{L}^{-1}\left\{\frac{2!}{(s-1)^{2+1}}\right\} = 2 \mathcal{L}^{-1}\left\{\frac{2!}{(s-1)^{2+1}}\right\} = 2 e^t t^2$$

8. Determine the inverse Laplace transforms of : (a) $\frac{1}{(s+2)^4}$ (b) $\frac{3}{(s-3)^5}$

$$(a) \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^{3+1}}\right\} = \frac{1}{3!} \mathcal{L}^{-1}\left\{\frac{3!}{(s+2)^{3+1}}\right\} = \frac{1}{6} \mathcal{L}^{-1}\left\{\frac{3!}{(s+2)^{3+1}}\right\} = \frac{1}{6} e^{-2t} t^3$$

$$(b) \mathcal{L}^{-1}\left\{\frac{3}{(s-3)^5}\right\} = 3 \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^{4+1}}\right\} = \frac{3}{4!} \mathcal{L}^{-1}\left\{\frac{4!}{(s-3)^{4+1}}\right\} = \frac{1}{8} \mathcal{L}^{-1}\left\{\frac{4!}{(s-3)^{4+1}}\right\} = \frac{1}{8} e^{3t} t^4$$

9. Determine the inverse Laplace transforms of : (a) $\frac{s+1}{s^2+2s+10}$ (b) $\frac{3}{s^2+6s+13}$

$$(a) \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\} = e^{-t} \cos 3t$$

$$(b) \mathcal{L}^{-1}\left\{\frac{3}{s^2+6s+13}\right\} = 3 \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+2^2}\right\} = \frac{3}{2} \mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+2^2}\right\} = \frac{3}{2} e^{-3t} \sin 2t$$

10. Determine the inverse Laplace transforms of : (a) $\frac{2(s-3)}{s^2-6s+13}$ (b) $\frac{7}{s^2-8s+12}$

$$(a) \mathcal{L}^{-1}\left\{\frac{2(s-3)}{s^2-6s+13}\right\} = 2 \mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+2^2}\right\} = 2e^{3t} \cos 2t$$

$$(b) \mathcal{L}^{-1}\left\{\frac{7}{s^2-8s+12}\right\} = 7 \mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2-2^2}\right\} = \frac{7}{2} \mathcal{L}^{-1}\left\{\frac{2}{(s-4)^2-2^2}\right\} = \frac{7}{2} e^{4t} \sinh 2t$$

11. Determine the inverse Laplace transforms of : (a) $\frac{2s+5}{s^2+4s-5}$ (b) $\frac{3s+2}{s^2-8s+25}$

$$(a) \mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+4s-5}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s+2)}{(s+2)^2-3^2} + \frac{1}{(s+2)^2-3^2}\right\}$$

$$= 2 \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2-3^2}\right\} + \frac{1}{3} \mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2-3^2}\right\} = 2e^{-2t} \cosh 3t + \frac{1}{3} e^{-2t} \sinh 3t$$

$$(b) \mathcal{L}^{-1}\left\{\frac{3s+2}{s^2-8s+25}\right\} = \mathcal{L}^{-1}\left\{\frac{3s+2}{(s-4)^2+3^2}\right\} = \mathcal{L}^{-1}\left\{\frac{3(s-4)+14}{(s-4)^2+3^2}\right\}$$

$$= 3 \mathcal{L}^{-1}\left\{\frac{s-4}{(s-4)^2+3^2}\right\} + \frac{14}{3} \mathcal{L}^{-1}\left\{\frac{3}{(s-4)^2+3^2}\right\} = 3e^{4t} \cos 3t + \frac{14}{3} e^{4t} \sin 3t$$

EXERCISE 252 Page 741

1. Use partial fractions to find the inverse Laplace transforms of: $\frac{11-3s}{s^2+2s-3}$

$$\frac{11-3s}{s^2+2s-3} = \frac{2}{(s-1)} - \frac{5}{(s+3)} \quad \text{from Problem 1, page 16 of textbook}$$

$$\text{Hence, } \mathcal{L}^{-1}\left\{\frac{11-3s}{s^2+2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{(s-1)} - \frac{5}{(s+3)}\right\} = 2e^{-t} - 5e^{-3t}$$

2. Use partial fractions to find the inverse Laplace transforms of: $\frac{2s^2-9s-35}{(s+1)(s-2)(s+3)}$

$$\frac{2s^2-9s-35}{(s+1)(s-2)(s+3)} = \frac{4}{(s+1)} - \frac{3}{(s-2)} + \frac{1}{(s+3)} \quad \text{from Problem 2, page 16 of textbook}$$

$$\text{Hence, } \mathcal{L}^{-1}\left\{\frac{2s^2-9s-35}{(s+1)(s-2)(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{(s+1)} - \frac{3}{(s-2)} + \frac{1}{(s+3)}\right\} = 4e^{-t} - 3e^{2t} + e^{-3t}$$

3. Use partial fractions to find the inverse Laplace transforms of: $\frac{5s^2-2s-19}{(s+3)(s-1)^2}$

$$\frac{5s^2-2s-19}{(s+3)(s-1)^2} = \frac{2}{(s+3)} + \frac{3}{(s-1)} - \frac{4}{(s-1)^2} \quad \text{from Problem 6, page 18 of textbook}$$

$$\text{Hence, } \mathcal{L}^{-1}\left\{\frac{5s^2-2s-19}{(s+3)(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{(s+3)} + \frac{3}{(s-1)} - \frac{4}{(s-1)^2}\right\} = 2e^{-3t} + 3e^t - 4e^t t$$

$$\text{Note: } \mathcal{L}^{-1}\left\{\frac{4}{(s-1)^2}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^{1+1}}\right\} = 4e^t t$$

4. Use partial fractions to find the inverse Laplace transforms of: $\frac{3s^2+16s+15}{(s+3)^3}$

$$\frac{3s^2 + 16s + 15}{(s+3)^3} = \frac{3}{(s+3)} - \frac{2}{(s+3)^2} - \frac{6}{(s+3)^3} \quad \text{from Problem 7, page 19 of textbook}$$

$$\begin{aligned} \text{Hence, } \mathcal{L}^{-1}\left\{\frac{3s^2 + 16s + 15}{(s+3)^3}\right\} &= \mathcal{L}^{-1}\left\{\frac{3}{(s+3)} - \frac{2}{(s+3)^2} - \frac{6}{(s+3)^3}\right\} \\ &= 3\mathcal{L}^{-1}\left\{\frac{1}{(s+3)}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} - 6\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^3}\right\} \\ &= 3\mathcal{L}^{-1}\left\{\frac{1}{(s+3)}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^{1+1}}\right\} - \frac{6}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{(s+3)^{2+1}}\right\} \\ &= 3e^{-3t} - 2e^{-3t}t - 3e^{-3t}t^2 \quad \text{or} \quad e^{-3t}(3 - 2t - 3t^2) \end{aligned}$$

5. Use partial fractions to find the inverse Laplace transforms of: $\frac{7s^2 + 5s + 13}{(s^2 + 2)(s+1)}$

$$\frac{7s^2 + 5s + 13}{(s^2 + 2)(s+1)} = \frac{2s+3}{(s^2 + 2)} + \frac{5}{(s+1)} \quad \text{from Problem 8, page 20 of textbook}$$

$$\text{and } \frac{2s+3}{(s^2 + 2)} + \frac{5}{(s+1)} = \frac{2s}{(s^2 + 2)} + \frac{3}{(s^2 + 2)} + \frac{5}{(s+1)}$$

$$\begin{aligned} \text{Hence, } \mathcal{L}^{-1}\left\{\frac{7s^2 + 5s + 13}{(s^2 + 2)(s+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{2s}{(s^2 + 2)} + \frac{3}{(s^2 + 2)} + \frac{5}{s+1}\right\} \\ &= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2 + (\sqrt{2})^2}\right\} + \frac{3}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2 + (\sqrt{2})^2}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= 2\cos\sqrt{2}t + \frac{3}{\sqrt{2}}\sin\sqrt{2}t + 5e^{-t} \end{aligned}$$

6. Use partial fractions to find the inverse Laplace transforms of: $\frac{3 + 6s + 4s^2 - 2s^3}{s^2(s^2 + 3)}$

$$\frac{3 + 6s + 4s^2 - 2s^3}{s^2(s^2 + 3)} = \frac{2}{s} + \frac{1}{s^2} + \frac{3-4s}{s^2 + 3} \quad \text{from Problem 9, page 20 of textbook}$$

$$\text{Hence, } \mathcal{L}^{-1}\left\{\frac{3 + 6s + 4s^2 - 2s^3}{s^2(s^2 + 3)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s} + \frac{1}{s^2} + \frac{3-4s}{s^2 + 3}\right\}$$

$$\begin{aligned}
&= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{3-4s}{s^2+(\sqrt{3})^2}\right\} \\
&= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s^2+(\sqrt{3})^2}\right\} - \mathcal{L}^{-1}\left\{\frac{4s}{s^2+(\sqrt{3})^2}\right\} \\
&= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{3}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{s^2+(\sqrt{3})^2}\right\} - 4\mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{3})^2}\right\} \\
&= \mathbf{2 + t + \sqrt{3} \sin \sqrt{3} t - 4 \cos \sqrt{3} t}
\end{aligned}$$

7. Use partial fractions to find the inverse Laplace transforms of: $\frac{26-s^2}{s(s^2+4s+13)}$

$$\text{Let } \frac{26-s^2}{s(s^2+4s+13)} \equiv \frac{A}{s} + \frac{Bs+C}{s^2+4s+13} = \frac{A(s^2+4s+13) + (Bs+C)s}{s(s^2+4s+13)}$$

$$\text{Hence, } 26-s^2 = A(s^2+4s+13) + Bs^2 + Cs$$

$$\text{When } s = 0: \quad 26 = 13A + 0 + 0 \quad \text{i.e. } \mathbf{A = 2}$$

$$\text{Equating } s^2 \text{ coefficients: } \quad -1 = A + B \quad \text{i.e. } \mathbf{B = -3}$$

$$\text{Equating } s \text{ coefficients: } \quad 0 = 4A + C \quad \text{i.e. } \mathbf{C = -8}$$

$$\text{Thus, } \frac{26-s^2}{s(s^2+4s+13)} \equiv \frac{2}{s} + \frac{-3s-8}{s^2+4s+13}$$

$$\begin{aligned}
\text{Hence, } \mathcal{L}^{-1}\left\{\frac{26-s^2}{s(s^2+4s+13)}\right\} &= \mathcal{L}^{-1}\left\{\frac{2}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{3s+8}{s^2+4s+13}\right\} \\
&= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{3s+8}{(s+2)^2+3^2}\right\} \\
&= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{3(s+2)}{(s+2)^2+3^2}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+3^2}\right\}
\end{aligned}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 3\mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2+3^2}\right\} - \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\}$$

$$= 2 - 3e^{-2t} \cos 3t - \frac{2}{3}e^{-2t} \sin 3t$$

EXERCISE 253 Page 743

1. Determine for the transfer function: $R(s) = \frac{50(s+4)}{s(s+2)(s^2-8s+25)}$

(a) the zero and (b) the poles. Show the poles and zeros on a pole-zero diagram.

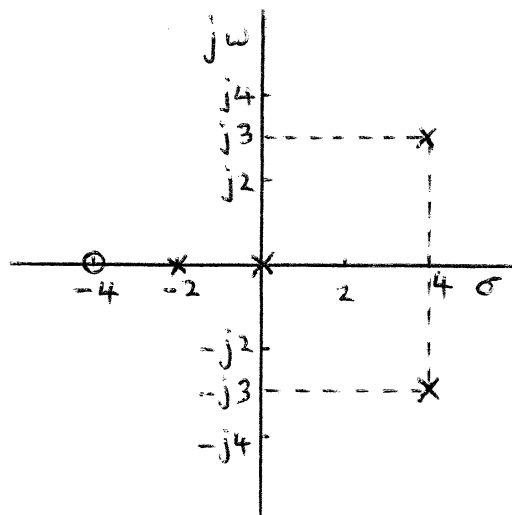
(a) For the numerator to be zero, $(s+4) = 0$ hence, **$s = -4$ is a zero of $R(s)$**

(b) For the denominator to be zero, $s = 0$ or $s = -2$ or $s^2 - 8s + 25 = 0$

$$\text{i.e. } s = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(25)}}{2(1)} = \frac{8 \pm \sqrt{-36}}{2} = \frac{-8 \pm j6}{2} = 4 \pm j3$$

Hence, **poles occur at $s = 0$, $s = -2$, $s = 4 + j3$ and $4 - j3$**

A pole-zero diagram is shown below.



2. Determine the poles and zeros for the function: $F(s) = \frac{(s-1)(s+2)}{(s+3)(s^2-2s+5)}$ and plot them on a pole-zero map.

For the numerator to be zero, $(s-1) = 0$ hence, **$s = +1$ is a zero of $F(s)$**

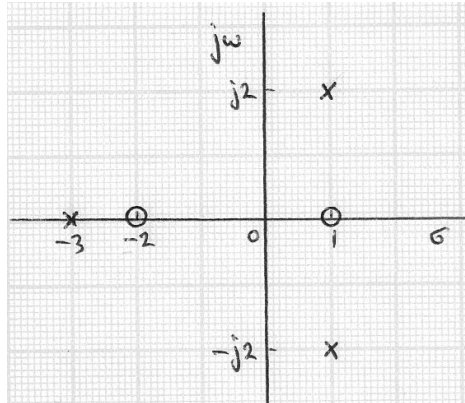
and $(s+2) = 0$ hence, **$s = -2$ is a zero of $F(s)$**

For the denominator to be zero, $s = -3$ or $s^2 - 2s + 5 = 0$

$$\text{i.e. } s = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm j4}{2} = 1 \pm j2$$

Hence, **poles occur at $s = -3$, $s = 1 + j2$ and $1 - j2$**

A pole-zero diagram is shown below.



3. For the function $G(s) = \frac{s-1}{(s+2)(s^2+2s+5)}$, determine the poles and zeros and show them on a pole-zero diagram.

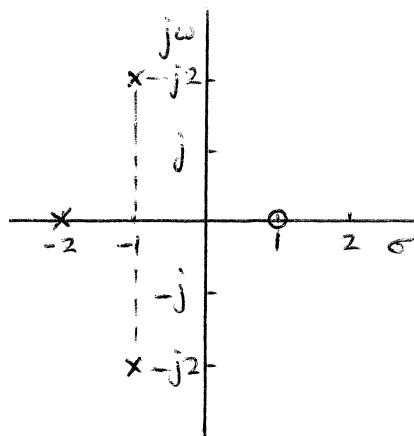
For the denominator to be zero, $s = -2$ or $s^2 + 2s + 5 = 0$

$$\text{i.e. } s = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm j4}{2} = -1 \pm j2$$

Hence, **poles occur at $s = -2$, $s = -1 + j2$ and $-1 - j2$**

For the numerator to be zero, $(s - 1) = 0$ hence, **$s = 1$ is a zero of $G(s)$**

A pole-zero diagram is shown below.



4. Find the poles and zeros for the transfer function: $H(s) = \frac{s^2 - 5s - 6}{s(s^2 + 4)}$ and plot the results in the s-plane.

For the denominator to be zero, $s = 0$ or $s^2 + 4 = 0$ and

i.e. $s^2 = -4$ and $s = \sqrt{-4} = \pm j2$

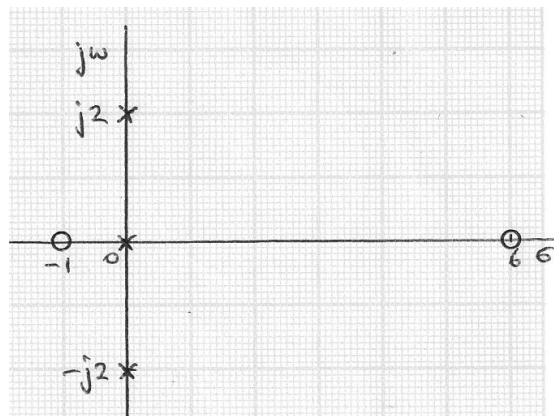
Hence, **poles occur at $s = 0$, $s = +j2$ and $-j2$**

For the numerator to be zero, $s^2 - 5s - 6 = 0$

i.e. $(s - 6)(s + 1) = 0$ hence, **$s = 6$ is a zero of $H(s)$**

and **$s = -1$ is a zero of $H(s)$**

A pole-zero diagram is shown below.



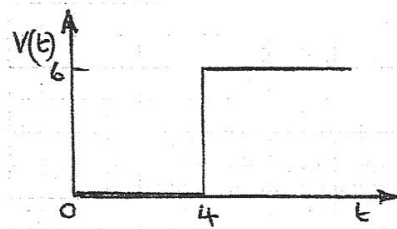
CHAPTER 70 THE LAPLACE TRANSFORM OF THE HEAVISIDE FUNCTION

EXERCISE 254 Page 747

1. A 6 V source is switched on at time $t = 4$ s. Write the function in terms of the Heaviside step function and sketch the waveform.

The function is shown sketched below.

The Heaviside step function is: $V(t) = 6 H(t - 4)$

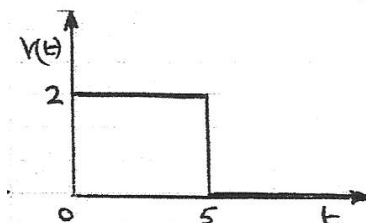


2. Write the function $V(t) = \begin{cases} 2 & \text{for } 0 < t < 5 \\ 0 & \text{for } t > 5 \end{cases}$ in terms of the Heaviside step function and sketch the waveform.

The voltage has a value of 2 up until time $t = 5$; then it is turned off.

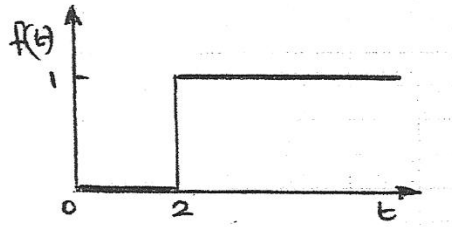
The function is shown sketched below.

The Heaviside step function is: $V(t) = H(t) - H(t - 5)$



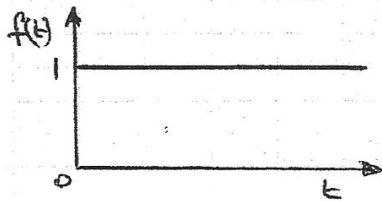
3. Sketch the graph of: $f(t) = H(t - 2)$

A function $H(t - 2)$ has a maximum value of 1 and starts when $t = 2$, as shown in the sketch below.



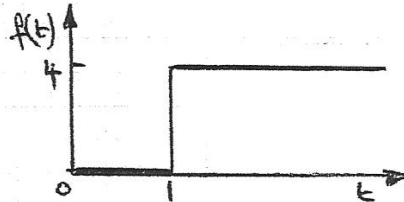
4. Sketch the graph of: $f(t) = H(t)$

A function $H(t)$ has a maximum value of 1 and starts when $t = 0$, as shown in the sketch below.



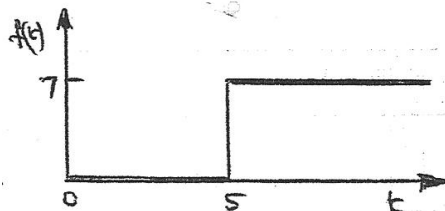
5. Sketch the graph of: $f(t) = 4H(t - 1)$

A function $4H(t - 1)$ has a maximum value of 4 and starts when $t = 1$, as shown in the sketch below.



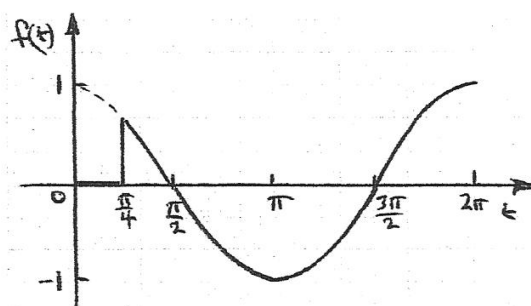
6. Sketch the graph of: $f(t) = 7H(t - 5)$

A function $7H(t - 5)$ has a maximum value of 7 and starts when $t = 5$, as shown in the sketch below.



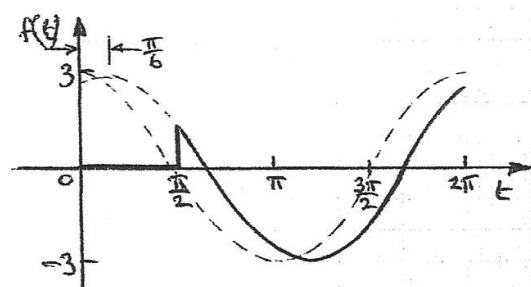
7. Sketch the graph of: $f(t) = H\left(t - \frac{\pi}{4}\right) \cdot \cos t$

Below shows a graph of $H\left(t - \frac{\pi}{4}\right) \cdot \cos t$ where the graph of $\cos t$ does not ‘switch on’ until $t = \pi/4$



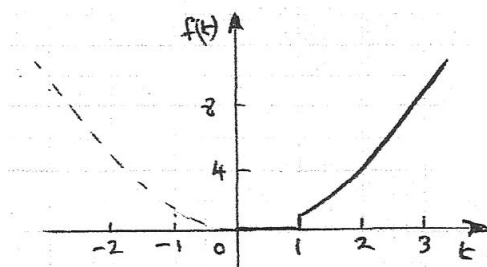
8. Sketch the graph of: $f(t) = 3H\left(t - \frac{\pi}{2}\right) \cdot \cos\left(t - \frac{\pi}{6}\right)$

Below shows a graph of $f(t) = 3H\left(t - \frac{\pi}{2}\right) \cdot \cos\left(t - \frac{\pi}{6}\right)$ where the graph of $3 \cos(t - \pi/6)$ does not ‘switch on’ until $t = \pi/2$



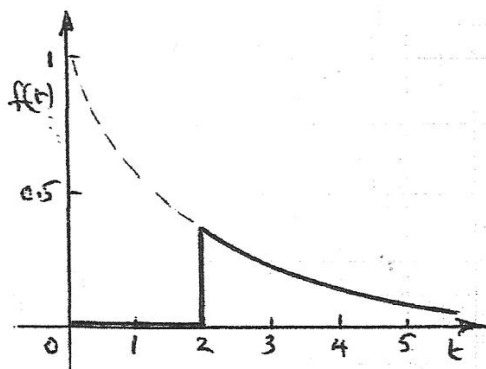
9. Sketch the graph of: $f(t) = H(t - 1) \cdot t^2$

Below shows a graph of $f(t) = H(t - 1) \cdot t^2$ where the graph of t^2 does not ‘switch on’ until $t = 1$



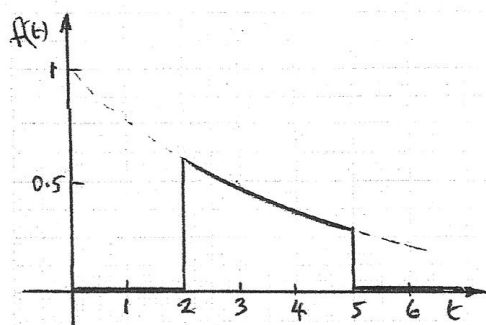
10. Sketch the graph of: $f(t) = H(t - 2) \cdot e^{-\frac{t}{2}}$

Below shows a graph of $f(t) = H(t - 2) \cdot e^{-\frac{t}{2}}$ where the graph of $e^{-\frac{t}{2}}$ does not ‘switch on’ until $t = 2$



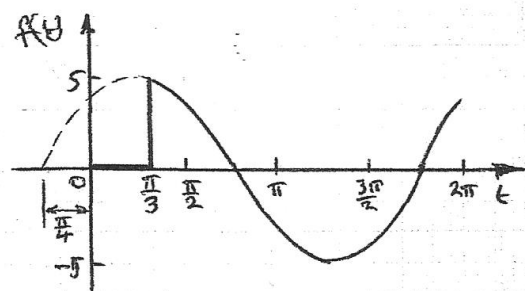
11. Sketch the graph of: $f(t) = [H(t - 2) - H(t - 5)] \cdot e^{-\frac{t}{4}}$

Below shows the graph of $f(t) = [H(t - 2) - H(t - 5)] \cdot e^{-\frac{t}{4}}$ where the graph of $e^{-\frac{t}{4}}$ does not ‘switch on’ until $t = 2$, but then ‘switches off’ at $t = 5$.



12. Sketch the graph of: $f(t) = 5H\left(t - \frac{\pi}{3}\right) \cdot \sin\left(t + \frac{\pi}{4}\right)$

Below shows a graph of $f(t) = 5H\left(t - \frac{\pi}{3}\right) \cdot \sin\left(t + \frac{\pi}{4}\right)$ where the graph of $5 \sin(t + \pi/4)$ does not ‘switch on’ until $t = \pi/3$



EXERCISE 255 Page 749

1. Determine $\mathcal{L}\{H(t-1)\}$

$$\mathcal{L}\{H(t-c).f(t-c)\} = e^{-cs} F(s) \quad \text{where in this case, } F(s) = \mathcal{L}\{1\} \text{ and } c = 1$$

Hence,
$$\mathcal{L}\{H(t-1)\} = e^{-s} \left(\frac{1}{s} \right) \quad \text{from (i) of Table 67.1, page 728}$$

$$= \frac{e^{-s}}{s}$$

2. Determine $\mathcal{L}\{7 H(t-3)\}$

$$\mathcal{L}\{H(t-c).f(t-c)\} = e^{-cs} F(s) \quad \text{where in this case, } F(s) = \mathcal{L}\{7\} \text{ and } c = 3$$

Hence,
$$\mathcal{L}\{7 H(t-3)\} = e^{-3s} \left(\frac{7}{s} \right) \quad \text{from (ii) of Table 67.1, page 728}$$

$$= \frac{7e^{-3s}}{s}$$

3. Determine $\mathcal{L}\{H(t-2).(t-2)^2\}$

$$\mathcal{L}\{H(t-c).f(t-c)\} = e^{-cs} F(s) \quad \text{where in this case, } F(s) = \mathcal{L}\{t^2\} \text{ and } c = 2$$

Hence,
$$\mathcal{L}\{H(t-2).f(t-2)^2\} = e^{-2s} \left(\frac{2!}{s^3} \right) \quad \text{from (vii) of Table 67.1, page 728}$$

$$= \frac{2e^{-2s}}{s^3}$$

4. Determine $\mathcal{L}\{H(t-3).\sin(t-3)\}$

$$\mathcal{L}\{H(t-c).f(t-c)\} = e^{-cs} F(s) \quad \text{where in this case, } F(s) = \mathcal{L}\{\sin t\} \text{ and } c = 3$$

Hence,
$$\mathcal{L}\{H(t-3).\sin(t-3)\} = e^{-3s} \left(\frac{1}{s^2+1^2} \right) \quad \text{from (iv) of Table 67.1, page 728}$$

$$= \frac{e^{-3s}}{s^2+1}$$

5. Determine $\mathcal{L}\{H(t-4).e^{t-4}\}$

$$\mathcal{L}\{H(t-c).f(t-c)\} = e^{-cs}F(s) \quad \text{where in this case, } F(s) = \mathcal{L}\{e^t\} \text{ and } c = 4$$

Hence, $\mathcal{L}\{H(t-4).e^{t-4}\} = e^{-4s}\left(\frac{1}{s-1}\right)$ from (iii) of Table 67.1, page 728

$$= \frac{e^{-4s}}{s-1}$$

6. Determine $\mathcal{L}\{H(t-5).\sin 3(t-5)\}$

$$\mathcal{L}\{H(t-c).f(t-c)\} = e^{-cs}F(s) \quad \text{where in this case, } F(s) = \mathcal{L}\{\sin 3t\} \text{ and } c = 5$$

Hence, $\mathcal{L}\{H(t-5).\sin 3(t-5)\} = e^{-5s}\left(\frac{3}{s^2+3^2}\right)$ from (iv) of Table 67.1, page 728

$$= \frac{3e^{-5s}}{s^2+9}$$

7. Determine $\mathcal{L}\{H(t-1).(t-1)^3\}$

$$\mathcal{L}\{H(t-c).f(t-c)\} = e^{-cs}F(s) \quad \text{where in this case, } F(s) = \mathcal{L}\{t^3\} \text{ and } c = 1$$

Hence, $\mathcal{L}\{H(t-1).(t-1)^3\} = e^{-s}\left(\frac{3!}{s^{3+1}}\right)$ from (viii) of Table 67.1, page 728

$$= \frac{6e^{-s}}{s^4}$$

8. Determine $\mathcal{L}\{H(t-6).\cos 3(t-6)\}$

$$\mathcal{L}\{H(t-c).f(t-c)\} = e^{-cs}F(s) \quad \text{where in this case, } F(s) = \mathcal{L}\{\cos 3t\} \text{ and } c = 6$$

Hence, $\mathcal{L}\{H(t-6).\cos 3(t-6)\} = e^{-6s}\left(\frac{s}{s^2+3^2}\right)$ from (v) of Table 67.1, page 728

$$= \frac{se^{-6s}}{s^2+9}$$

9. Determine $\mathcal{L}\{5H(t-5).\sinh 2(t-5)\}$

$$\mathcal{L}\{H(t-c).f(t-c)\} = e^{-cs} F(s) \quad \text{where in this case, } F(s) = \mathcal{L}\{\sinh 2t\} \text{ and } c = 5$$

$$\text{Hence, } \mathcal{L}\{5 H(t-5).\sinh 2(t-5)\} = 5 e^{-5s} \left(\frac{2}{s^2 - 2^2} \right) \quad \text{from (x) of Table 67.1, page 728}$$

$$= \frac{10 e^{-5s}}{s^2 - 4}$$

$$\mathbf{10. \text{ Determine } \mathcal{L}\left\{H\left(t - \frac{\pi}{3}\right).\cos 2\left(t - \frac{\pi}{3}\right)\right\}}$$

$$\mathcal{L}\{H(t-c).f(t-c)\} = e^{-cs} F(s) \quad \text{where in this case, } F(s) = \mathcal{L}\{\cos 2t\} \text{ and } c = \frac{\pi}{3}$$

$$\text{Hence, } \mathcal{L}\left\{\left(t - \frac{\pi}{3}\right).\cos 2\left(t - \frac{\pi}{3}\right)\right\} = e^{-\frac{\pi}{3}s} \left(\frac{s}{s^2 + 2^2} \right) \quad \text{from (v) of Table 67.1, page 728}$$

$$= \frac{s e^{-\frac{\pi}{3}s}}{s^2 + 4}$$

$$\mathbf{11. \text{ Determine } \mathcal{L}\{2 H(t-3).e^{t-3}\}}$$

$$\mathcal{L}\{H(t-c).f(t-c)\} = e^{-cs} F(s) \quad \text{where in this case, } F(s) = \mathcal{L}\{e^t\} \text{ and } c = 3$$

$$\text{Hence, } \mathcal{L}\{2 H(t-3).e^{t-3}\} = 2 e^{-3s} \left(\frac{1}{s-1} \right) \quad \text{from (iii) of Table 67.1, page 728}$$

$$= \frac{2 e^{-3s}}{s-1}$$

$$\mathbf{12. \text{ Determine } \mathcal{L}\{3 H(t-2).\cosh(t-2)\}}$$

$$\mathcal{L}\{H(t-c).f(t-c)\} = e^{-cs} F(s) \quad \text{where in this case, } F(s) = \mathcal{L}\{\cosh t\} \text{ and } c = 2$$

$$\text{Hence, } \mathcal{L}\{3 H(t-2).\cosh(t-2)\} = 3 e^{-2s} \left(\frac{s}{s^2 - 1^2} \right) \quad \text{from (ix) of Table 67.1, page 728}$$

$$= \frac{3 s e^{-2s}}{s^2 - 1}$$

EXERCISE 256 Page 750

$$1. \text{ Determine } \mathcal{L}^{-1} \left\{ \frac{e^{-9s}}{s} \right\}$$

Part of the numerator corresponds to e^{-cs} where $c = 9$. This indicates $H(t - 9)$

Then $\frac{1}{s} = F(s) = \mathcal{L}\{1\}$ from (i) of Table 67.1, page 728

Hence, $\mathcal{L}^{-1} \left\{ \frac{e^{-9s}}{s} \right\} = H(t - 9)$

$$2. \text{ Determine } \mathcal{L}^{-1} \left\{ \frac{4e^{-3s}}{s} \right\}$$

Part of the numerator corresponds to e^{-cs} where $c = 3$. This indicates $H(t - 3)$

Then $\frac{4}{s} = F(s) = \mathcal{L}\{4\}$ from (ii) of Table 67.1, page 728

Hence, $\mathcal{L}^{-1} \left\{ \frac{4e^{-3s}}{s} \right\} = 4 H(t - 3)$

$$3. \text{ Determine } \mathcal{L}^{-1} \left\{ \frac{2e^{-2s}}{s^2} \right\}$$

The numerator corresponds to e^{-cs} where $c = 2$. This indicates $H(t - 2)$

$$\frac{1}{s^2} = F(s) = \mathcal{L}\{t\} \text{ from (vii) of Table 67.1, page 728}$$

Then $\mathcal{L}^{-1} \left\{ \frac{2e^{-2s}}{s^2} \right\} = 2 H(t - 2) \cdot (t - 2)$

$$4. \text{ Determine } \mathcal{L}^{-1} \left\{ \frac{5e^{-2s}}{s^2 + 1} \right\}$$

Part of the numerator corresponds to e^{-cs} where $c = 2$. This indicates $H(t - 2)$

$$\frac{5}{s^2+1} \text{ may be written as: } 5\left(\frac{1}{s^2+1^2}\right)$$

$$\text{Then } 5\left(\frac{1}{s^2+1^2}\right) = F(s) = \mathcal{L}\{5 \sin t\} \text{ from (iv) of Table 67.1, page 728}$$

$$\text{Hence, } \mathcal{L}^{-1}\left\{\frac{5e^{-2s}}{s^2+1}\right\} = H(t-2).5 \sin(t-2) = 5 H(t-2).\sin(t-2)$$

$$\mathbf{5. \text{ Determine } \mathcal{L}^{-1}\left\{\frac{3s e^{-4s}}{s^2+16}\right\}}$$

Part of the numerator corresponds to e^{-cs} where $c = 4$. This indicates $H(t-4)$

$$\frac{3s}{s^2+16} \text{ may be written as: } 3\left(\frac{s}{s^2+4^2}\right)$$

$$\text{Then } 3\left(\frac{s}{s^2+4^2}\right) = F(s) = \mathcal{L}\{3 \cos 4t\} \text{ from (v) of Table 67.1, page 728}$$

$$\text{Hence, } \mathcal{L}^{-1}\left\{\frac{3s e^{-4s}}{s^2+4^2}\right\} = H(t-4).3 \cos 4(t-4) = 3 H(t-4).\cos 4(t-4)$$

$$\mathbf{6. \text{ Determine } \mathcal{L}^{-1}\left\{\frac{6e^{-2s}}{s^2-1}\right\}}$$

Part of the numerator corresponds to e^{-cs} where $c = 2$. This indicates $H(t-2)$

$$\frac{6}{s^2-1} \text{ may be written as: } 6\left(\frac{1}{s^2-1^2}\right)$$

$$\text{Then } 6\left(\frac{1}{s^2-1^2}\right) = F(s) = \mathcal{L}\{6 \sinh t\} \text{ from (x) of Table 67.1, page 728}$$

$$\text{Hence, } \mathcal{L}^{-1}\left\{\frac{6e^{-2s}}{s^2-1}\right\} = H(t-2).6 \sinh(t-2) = 6 H(t-2).\sinh(t-2)$$

$$\mathbf{7. \text{ Determine } \mathcal{L}^{-1}\left\{\frac{3e^{-6s}}{s^3}\right\}}$$

The numerator corresponds to e^{-cs} where $c = 6$. This indicates $H(t-6)$

$$\frac{1}{s^3} = F(s) = \mathcal{L}\left\{\frac{1}{2}t^2\right\} \quad \text{from (viii) of Table 67.1, page 728}$$

Then
$$\mathcal{L}^{-1}\left\{\frac{3e^{-6s}}{s^2}\right\} = 3 H(t-6) \cdot \frac{1}{2}(t-6)^2 = 1.5 H(t-6) \cdot (t-6)^2$$

8. Determine $\mathcal{L}^{-1}\left\{\frac{2s e^{-4s}}{s^2-16}\right\}$

Part of the numerator corresponds to e^{-cs} where $c = 4$. This indicates $H(t-4)$

$$\frac{2s}{s^2-16} \quad \text{may be written as: } 2\left(\frac{s}{s^2-4^2}\right)$$

Then
$$2\left(\frac{s}{s^2-4^2}\right) = F(s) = \mathcal{L}\{2 \cosh 4t\} \quad \text{from (ix) of Table 67.1, page 728}$$

Hence,
$$\mathcal{L}^{-1}\left\{\frac{2s e^{-4s}}{s^2-4^2}\right\} = H(t-4) \cdot 2 \cosh 4(t-4) = 2 H(t-4) \cdot \cosh 4(t-4)$$

9. Determine $\mathcal{L}^{-1}\left\{\frac{2s e^{-\frac{1}{2}s}}{s^2+5}\right\}$

Part of the numerator corresponds to e^{-cs} where $c = \frac{1}{2}$. This indicates $H\left(t - \frac{1}{2}\right)$

$$\frac{2s}{s^2+5} \quad \text{may be written as: } 2\left(\frac{s}{s^2+(\sqrt{5})^2}\right)$$

Then
$$2\left(\frac{s}{s^2+(\sqrt{5})^2}\right) = F(s) = \mathcal{L}\{2 \cos \sqrt{5} t\} \quad \text{from (v) of Table 67.1, page 728}$$

Hence,
$$\mathcal{L}^{-1}\left\{\frac{2s e^{-\frac{1}{2}s}}{s^2+(\sqrt{5})^2}\right\} = H\left(t - \frac{1}{2}\right) \cdot 2 \cos \sqrt{5}\left(t - \frac{1}{2}\right) = 2 H\left(t - \frac{1}{2}\right) \cdot \cos \sqrt{5}\left(t - \frac{1}{2}\right)$$

10. Determine $\mathcal{L}^{-1}\left\{\frac{4 e^{-7s}}{s-1}\right\}$

Part of the numerator corresponds to e^{-cs} where $c = 7$. This indicates $H(t - 7)$

Then $\frac{1}{s-1} = F(s) = \mathcal{L}\{e^t\}$ from (iii) of Table 67.1, page 728

Hence, $\mathcal{L}^{-1}\left\{\frac{4e^{-7s}}{s-1}\right\} = 4 H(t-7) \cdot e^{t-7}$

CHAPTER 71 THE SOLUTION OF DIFFERENTIAL EQUATIONS USING LAPLACE TRANSFORM

EXERCISE 257 Page 755

1. A first order differential equation involving current i in a series R-L circuit is given by:

$$\frac{di}{dt} + 5i = \frac{E}{2} \quad \text{and } i = 0 \text{ at time } t = 0$$

Use Laplace transforms to solve for i when (a) $E = 20$ (b) $E = 40e^{-3t}$ and (c) $E = 50 \sin 5t$

Taking the Laplace transform of each term of $\frac{di}{dt} + 5i = \frac{E}{2}$ gives:

$$\mathcal{L}\left\{\frac{di}{dt}\right\} + 5\mathcal{L}\{i\} = \mathcal{L}\left\{\frac{E}{2}\right\}$$

i.e.
$$s\mathcal{L}\{i\} - i(0) + 5\mathcal{L}\{i\} = \frac{E/2}{s}$$

$i = 0$ at $t = 0$, hence, $i(0) = 0$

Hence,
$$(s + 5)\mathcal{L}\{i\} = \frac{E/2}{s}$$

i.e.
$$\mathcal{L}\{i\} = \frac{E/2}{s(s+5)}$$

and
$$i = \mathcal{L}^{-1}\left\{\frac{E/2}{s(s+5)}\right\} = \frac{E}{2} \mathcal{L}^{-1}\left\{\frac{1}{s(s+5)}\right\}$$

Let
$$\frac{1}{s(s+5)} \equiv \frac{A}{s} + \frac{B}{(s+5)} = \frac{A(s+5) + Bs}{s(s+5)}$$

Hence,
$$1 = A(s+5) + Bs$$

When $s = 0$:
$$1 = 5A \quad \text{i.e.} \quad A = \frac{1}{5}$$

When $s = -5$:
$$1 = -5B \quad \text{i.e.} \quad B = -\frac{1}{5}$$

Thus,
$$i = \frac{E}{2} \mathcal{L}^{-1}\left\{\frac{1}{s(s+5)}\right\} = \frac{E}{2} \mathcal{L}^{-1}\left\{\frac{1}{5s} - \frac{1}{5(s+5)}\right\} = \frac{E}{2} \left(\frac{1}{5} - \frac{1}{5}e^{-5t}\right)$$

(a) When $E = 20$,
$$\mathbf{i} = \frac{20}{2} \left(\frac{1}{5} - \frac{1}{5} e^{-5t} \right) = 2(1 - e^{-5t})$$

(b) When $E = 40e^{-3t}$
$$\mathcal{L} \left\{ \frac{di}{dt} \right\} + 5\mathcal{L}\{i\} = \mathcal{L} \left\{ \frac{40e^{-3t}}{2} \right\}$$

i.e.
$$s\mathcal{L}\{i\} - i(0) + 5\mathcal{L}\{i\} = \frac{20}{s+3}$$

$i = 0$ at $t = 0$, hence, $i(0) = 0$

Hence,
$$(s+5)\mathcal{L}\{i\} = \frac{20}{s+3}$$

i.e.
$$\mathcal{L}\{i\} = \frac{20}{(s+3)(s+5)}$$

and
$$i = \mathcal{L}^{-1} \left\{ \frac{20}{(s+3)(s+5)} \right\}$$

Let
$$\frac{20}{(s+3)(s+5)} \equiv \frac{A}{(s+3)} + \frac{B}{(s+5)} = \frac{A(s+5) + B(s+3)}{(s+3)(s+5)}$$

Hence,
$$20 = A(s+5) + B(s+3)$$

When $s = -3$:
$$20 = 2A \quad \text{i.e.} \quad A = 10$$

When $s = -5$:
$$20 = -2B \quad \text{i.e.} \quad B = -10$$

Thus,
$$i = \mathcal{L}^{-1} \left\{ \frac{20}{(s+3)(s+5)} \right\} = \mathcal{L}^{-1} \left\{ \frac{10}{(s+3)} - \frac{10}{(s+5)} \right\}$$

i.e.
$$\mathbf{i} = 10e^{-3t} - 10e^{-5t} = 10(e^{-3t} - e^{-5t})$$

(c) When $E = 50 \sin 5t$
$$\mathcal{L} \left\{ \frac{di}{dt} \right\} + 5\mathcal{L}\{i\} = \mathcal{L} \left\{ \frac{50 \sin 5t}{2} \right\}$$

i.e.
$$s\mathcal{L}\{i\} - i(0) + 5\mathcal{L}\{i\} = \frac{25(5)}{s^2 + 5^2}$$

$i = 0$ at $t = 0$, hence, $i(0) = 0$

Hence,
$$(s+5)\mathcal{L}\{i\} = \frac{125}{s^2 + 25}$$

i.e.
$$\mathcal{L}\{i\} = \frac{125}{(s+5)(s^2 + 25)}$$

and
$$i = \mathcal{L}^{-1} \left\{ \frac{125}{(s+5)(s^2 + 25)} \right\}$$

Let
$$\frac{125}{(s+5)(s^2+25)} \equiv \frac{A}{(s+5)} + \frac{Bs+C}{(s^2+25)} = \frac{A(s^2+25) + (Bs+C)(s+5)}{(s+5)(s^2+25)}$$

Hence,
$$125 = A(s^2+25) + (Bs+C)(s+5)$$

When $s = -5$:
$$125 = 50A \quad \text{i.e.} \quad A = \frac{5}{2}$$

Equating s^2 coefficients:
$$0 = A + B \quad \text{i.e.} \quad B = -\frac{5}{2}$$

Equating constant terms:
$$125 = 25A + 5C \quad \text{i.e.} \quad 125 = \frac{125}{2} + 5C$$

from which,
$$C = \frac{\frac{125}{2}}{5} = \frac{25}{2}$$

Thus,
$$\begin{aligned} i &= \mathcal{L}^{-1} \left\{ \frac{125}{(s+5)(s^2+25)} \right\} = \mathcal{L}^{-1} \left\{ \frac{\frac{5}{2}}{(s+5)} + \frac{-\frac{5}{2}s + \frac{25}{2}}{(s^2+25)} \right\} \\ &= \frac{5}{2} \mathcal{L}^{-1} \left\{ \frac{1}{(s+5)} \right\} + \mathcal{L}^{-1} \left\{ \frac{\frac{25}{2}}{(s^2+25)} \right\} - \mathcal{L}^{-1} \left\{ \frac{\frac{5}{2}s}{(s^2+25)} \right\} \\ &= \frac{5}{2} \mathcal{L}^{-1} \left\{ \frac{1}{(s+5)} \right\} + \frac{5}{2} \mathcal{L}^{-1} \left\{ \frac{5}{(s^2+5^2)} \right\} - \frac{5}{2} \mathcal{L}^{-1} \left\{ \frac{s}{(s^2+5^2)} \right\} \\ &= \frac{5}{2} e^{-5t} + \frac{5}{2} \sin 5t - \frac{5}{2} \cos 5t \end{aligned}$$

i.e.
$$i = \frac{5}{2} (e^{-5t} + \sin 5t - \cos 5t)$$

2. Use Laplace transforms to solve the differential equation:

$$9 \frac{d^2 y}{dt^2} - 24 \frac{dy}{dt} + 16y = 0, \text{ given } y(0) = 3 \text{ and } y'(0) = 3$$

Taking Laplace transforms of each term of $9 \frac{d^2 y}{dt^2} - 24 \frac{dy}{dt} + 16y = 0$

gives:
$$9\mathcal{L} \left\{ \frac{dy^2}{dt^2} \right\} - 24\mathcal{L} \left\{ \frac{dy}{dt} \right\} + 16\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

Hence, $9[s^2 \mathcal{L}\{y\} - sy(0) - y'(0)] - 24[s\mathcal{L}\{y\} - y(0)] + 16\mathcal{L}\{y\} = 0$

Since $y(0) = 3$ and $y'(0) = 3$

then $9[s^2 \mathcal{L}\{y\} - 3s - 3] - 24[s\mathcal{L}\{y\} - 3] + 16\mathcal{L}\{y\} = 0$

i.e. $9s^2 \mathcal{L}\{y\} - 27s - 27 - 24s\mathcal{L}\{y\} + 72 + 16\mathcal{L}\{y\} = 0$

i.e. $(9s^2 - 24s + 16)\mathcal{L}\{y\} = 27s + 27 - 72$

and
$$\mathcal{L}\{y\} = \frac{27s - 45}{(9s^2 - 24s + 16)}$$

i.e.
$$y = \mathcal{L}^{-1}\left\{\frac{27s - 45}{9s^2 - 24s + 16}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{27s - 45}{(3s - 4)^2}\right\}$$

Let
$$\frac{27s - 45}{(3s - 4)^2} = \frac{A}{(3s - 4)} + \frac{B}{(3s - 4)^2} = \frac{A(3s - 4) + B}{(3s - 4)^2}$$

Hence, $27s - 45 = A(3s - 4) + B$

Let $s = 4/3$ $36 - 45 = B$ i.e. $B = -9$

Equating s coefficients gives: $27 = 3A$ i.e. $A = 9$

Hence,
$$\mathcal{L}^{-1}\left\{\frac{27s - 45}{(3s - 4)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{9}{(3s - 4)}\right\} - \mathcal{L}^{-1}\left\{\frac{9}{(3s - 4)^2}\right\}$$

i.e.
$$= \mathcal{L}^{-1}\left\{\frac{9}{3\left(s - \frac{4}{3}\right)}\right\} - \mathcal{L}^{-1}\left\{\frac{9}{3^2\left(s - \frac{4}{3}\right)^2}\right\}$$

i.e.
$$= \mathcal{L}^{-1}\left\{\frac{3}{\left(s - \frac{4}{3}\right)}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{\left(s - \frac{4}{3}\right)^2}\right\}$$

i.e. $y = 3e^{\frac{4}{3}} - e^{\frac{4}{3}}t$

i.e. $y = (3 - t)e^{\frac{4}{3}t}$

3. Use Laplace transforms to solve the differential equation:

$$\frac{d^2x}{dt^2} + 100x = 0, \text{ given } x(0) = 2 \text{ and } x'(0) = 0$$

$$\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\} + \mathcal{L}\{100x\} = \mathcal{L}\{0\}$$

i.e. $s^2 \mathcal{L}\{x\} - s x(0) - x'(0) + 100\mathcal{L}\{x\} = 0$

$x(0) = 2$ and $x'(0) = 0$, hence $s^2 \mathcal{L}\{x\} - 2s + 100\mathcal{L}\{x\} = 0$

i.e. $(s^2 + 100)\mathcal{L}\{x\} = 2s$

from which, $\mathcal{L}\{x\} = \frac{2s}{s^2 + 100}$ and $x = \mathcal{L}^{-1}\left\{\frac{2s}{s^2 + 100}\right\}$

i.e. $x = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 10^2}\right\} = 2 \cos 10t$

4. Use Laplace transforms to solve the differential equation:

$$\frac{d^2i}{dt^2} + 1000\frac{di}{dt} + 250000i = 0, \text{ given } i(0) = 0 \text{ and } i'(0) = 100$$

$$\mathcal{L}\left\{\frac{d^2i}{dt^2}\right\} + 1000\mathcal{L}\left\{\frac{di}{dt}\right\} + 250000\mathcal{L}\{i\} = \mathcal{L}\{0\}$$

i.e. $[s^2 \mathcal{L}\{i\} - s i(0) - i'(0)] + 1000[s\mathcal{L}\{i\} - i(0)] + 250000\mathcal{L}\{i\} = 0$

$i(0) = 0$ and $i'(0) = 100$, hence

$$s^2 \mathcal{L}\{i\} - 100 + 1000s\mathcal{L}\{i\} + 250000\mathcal{L}\{i\} = 0$$

i.e. $(s^2 + 1000s + 250000)\mathcal{L}\{i\} = 100$

from which, $\mathcal{L}\{i\} = \frac{100}{s^2 + 1000s + 250000}$

and $i = \mathcal{L}^{-1}\left\{\frac{100}{(s + 500)^2}\right\} = 100\mathcal{L}^{-1}\left\{\frac{1}{(s + 500)^{1+1}}\right\}$

i.e. $i = 100te^{-500t}$

5. Use Laplace transforms to solve the differential equation:

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0, \text{ given } x(0) = 4 \text{ and } x'(0) = 8$$

Taking Laplace transforms of each term of $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0$

gives: $\mathcal{L}\left\{\frac{dx^2}{dt^2}\right\} + 6\mathcal{L}\left\{\frac{dx}{dt}\right\} + 8\mathcal{L}\{x\} = \mathcal{L}\{0\}$

Hence, $[s^2 \mathcal{L}\{x\} - sx(0) - x'(0)] + 6[s\mathcal{L}\{x\} - x(0)] + 8\mathcal{L}\{x\} = 0$

Since $y(0) = 4$ and $y'(0) = 8$

then $[s^2 \mathcal{L}\{x\} - 4s - 8] + 6[s\mathcal{L}\{x\} - 4] + 8\mathcal{L}\{x\} = 0$

$$s^2 \mathcal{L}\{x\} - 4s - 8 + 6s\mathcal{L}\{x\} - 24 + 8\mathcal{L}\{x\} = 0$$

i.e. $(s^2 + 6s + 8)\mathcal{L}\{x\} = 4s + 8 + 24$

$$\mathcal{L}\{x\} = \frac{4s + 32}{(s^2 + 6s + 8)}$$

i.e. $y = \mathcal{L}^{-1}\left\{\frac{4s + 32}{s^2 + 6s + 8}\right\}$

Let $\frac{4s + 32}{(s^2 + 6s + 8)} = \frac{4s + 32}{(s + 2)(s + 4)} \equiv \frac{A}{(s + 2)} + \frac{B}{(s + 4)}$

$$= \frac{A(s + 4) + B(s + 2)}{(s + 2)(s + 4)}$$

i.e. $4s + 32 = A(s + 4) + B(s + 2)$

When $s = -2$: $24 = 2A$ from which, $A = 12$

When $s = -4$: $16 = -2B$ from which, $B = -8$

Hence, $y = \mathcal{L}^{-1}\left\{\frac{4s + 32}{s^2 + 6s + 8}\right\} = \mathcal{L}^{-1}\left\{\frac{12}{(s + 2)} - \frac{8}{(s + 4)}\right\}$

i.e. $y = 12e^{-2t} - 8e^{-4t}$ or $y = 4(3e^{-2t} - 2e^{-4t})$

6. Use Laplace transforms to solve the differential equation:

$$\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = 3e^{4x}, \text{ given } y(0) = -\frac{2}{3} \text{ and } y'(0) = 4\frac{1}{3}$$

$$\mathcal{L}\left\{\frac{d^2 y}{dx^2}\right\} - 2\mathcal{L}\left\{\frac{dy}{dx}\right\} + \mathcal{L}\{y\} = \mathcal{L}\{3e^{4x}\}$$

$$\text{i.e. } [s^2 \mathcal{L}\{y\} - s y(0) - y'(0)] - 2[s \mathcal{L}\{y\} - y(0)] + \mathcal{L}\{y\} = \frac{3}{(s-4)}$$

$$y(0) = -\frac{2}{3} \text{ and } y'(0) = 4\frac{1}{3}, \text{ hence}$$

$$s^2 \mathcal{L}\{y\} - s\left(-\frac{2}{3}\right) - \frac{13}{3} - 2s \mathcal{L}\{y\} + 2\left(-\frac{2}{3}\right) + \mathcal{L}\{y\} = \frac{3}{(s-4)}$$

$$\text{i.e. } (s^2 - 2s + 1)\mathcal{L}\{y\} + \frac{2}{3}s - \frac{13}{3} - \frac{4}{3} = \frac{3}{(s-4)}$$

$$\text{from which, } (s^2 - 2s + 1)\mathcal{L}\{y\} = \frac{3}{(s-4)} - \frac{2}{3}s + \frac{17}{3} = \frac{9 - 2s(s-4) + 17(s-4)}{3(s-4)}$$

$$\mathcal{L}\{y\} = \frac{9 - 2s^2 + 8s + 17s - 68}{3(s-4)(s^2 - 2s + 1)} = \frac{-59 + 25s - 2s^2}{3(s-4)(s-1)^2}$$

and

$$y = \mathcal{L}^{-1}\left\{\frac{-59 + 25s - 2s^2}{3(s-4)(s-1)^2}\right\}$$

$$\text{Let } \frac{\frac{1}{3}(-59 + 25s - 2s^2)}{(s-4)(s-1)^2} \equiv \frac{A}{(s-4)} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2} = \frac{A(s-1)^2 + B(s-4)(s-1) + C(s-4)}{(s-4)(s-1)^2}$$

$$\text{Hence, } -\frac{59}{3} + \frac{25}{3}s - \frac{2}{3}s^2 = A(s-1)^2 + B(s-4)(s-1) + C(s-4)$$

$$\text{When } s = 4: -\frac{59}{3} + \frac{100}{3} - \frac{32}{3} = 9A + 0 + 0 \quad \text{i.e. } 3 = 9A \quad \text{and} \quad A = \frac{1}{3}$$

$$\text{When } s = 1: -\frac{59}{3} + \frac{25}{3} - \frac{2}{3} = 0 + 0 - 3C \quad \text{i.e. } -12 = -3C \quad \text{and} \quad C = 4$$

$$\text{Equating } s^2 \text{ coefficients: } -\frac{2}{3} = A + B \quad \text{i.e. } -\frac{2}{3} = \frac{1}{3} + B \quad \text{and} \quad B = -1$$

$$\text{Hence, } y = \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{(s-4)} - \frac{1}{(s-1)} + \frac{4}{(s-1)^2}\right\}$$

i.e. $y = \frac{1}{3}e^{4x} - e^x + 4xe^x$ or $y = (4x-1)e^x + \frac{1}{3}e^{4x}$

7. Use Laplace transforms to solve the differential equation:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 3 \sin x, \text{ given } y(0) = 0 \text{ and } y'(0) = 0$$

$$\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} - 3\mathcal{L}\left\{\frac{dy}{dx}\right\} - 4\mathcal{L}\{y\} = \mathcal{L}\{3 \sin x\}$$

i.e. $[s^2\mathcal{L}\{y\} - sy(0) - y'(0)] - 3[s\mathcal{L}\{y\} - y(0)] - 4\mathcal{L}\{y\} = \frac{3}{s^2+1^2}$

$y(0) = 0$ and $y'(0) = 0$, hence

$$s^2\mathcal{L}\{y\} - 3s\mathcal{L}\{y\} - 4\mathcal{L}\{y\} = \frac{3}{s^2+1}$$

i.e. $(s^2 - 3s - 4)\mathcal{L}\{y\} = \frac{3}{s^2+1}$

from which, $\mathcal{L}\{y\} = \frac{3}{(s^2-3s-4)(s^2+1)} = \frac{3}{(s+1)(s-4)(s^2+1)}$

Let
$$\frac{3}{(s+1)(s-4)(s^2+1)} \equiv \frac{A}{(s+1)} + \frac{B}{(s-4)} + \frac{Cs+D}{(s^2+1)}$$

$$= \frac{A(s-4)(s^2+1) + B(s+1)(s^2+1) + (Cs+D)(s+1)(s-4)}{(s+1)(s-4)(s^2+1)}$$

i.e. $3 = A(s-4)(s^2+1) + B(s+1)(s^2+1) + (Cs+D)(s+1)(s-4)$

When $s = -1$: $3 = A(-5)(2)$ from which, $A = -\frac{3}{10}$

When $s = 4$: $3 = B(5)(17)$ from which, $B = \frac{3}{85}$

Equating s^3 coefficients gives: $0 + A + B + C$ i.e. $0 = -\frac{3}{10} + \frac{3}{85} + C$

and
$$C = \frac{3}{10} - \frac{3}{85} = \frac{51-6}{170} = \frac{45}{170}$$

Equating constants gives: $3 = -4A + B - 4D$ i.e. $4D = (4)\frac{3}{10} + \frac{3}{85} - 3 = -\frac{30}{17}$

and
$$D = -\frac{30}{68}$$

Hence,
$$y = \mathcal{L}^{-1}\left\{\frac{3}{(s+1)(s+4)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{-3/10}{(s+1)} + \frac{3/85}{(s-4)} + \frac{\frac{45}{170}s - \frac{30}{68}}{(s^2+1)}\right\}$$

and
$$y = \mathcal{L}^{-1}\left\{\frac{-3/10}{(s+1)} + \frac{3/85}{(s-4)} + \frac{\frac{45}{170}s}{(s^2+1)} - \frac{\frac{30}{68}}{(s^2+1)}\right\}$$

i.e.
$$y = -\frac{3}{10}e^{-x} + \frac{3}{85}e^{4x} + \frac{45}{170}\cos x - \frac{30}{68}\sin x$$

or
$$y = \frac{3}{85}e^{4x} - \frac{3}{10}e^{-x} + \frac{9}{34}\cos x - \frac{15}{34}\sin x$$

8. Use Laplace transforms to solve the differential equation:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 3 \cos 3x - 11 \sin 3x, \text{ given } y(0) = 0 \text{ and } y'(0) = 6$$

$$\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} + \mathcal{L}\left\{\frac{dy}{dx}\right\} - 2\mathcal{L}\{y\} = \mathcal{L}\{3 \cos 3x - 11 \sin 3x\}$$

i.e.
$$[s^2 \mathcal{L}\{y\} - s y(0) - y'(0)] + [s \mathcal{L}\{y\} - y(0)] - 2\mathcal{L}\{y\} = \frac{3s}{s^2+9} - \frac{33}{s^2+9}$$

$y(0) = 0$ and $y'(0) = 6$, hence

$$s^2 \mathcal{L}\{y\} - 6 + s \mathcal{L}\{y\} - 2\mathcal{L}\{y\} = \frac{3s-33}{s^2+9}$$

i.e.
$$(s^2 + s - 2)\mathcal{L}\{y\} = 6 + \frac{3s-33}{s^2+9}$$

from which,
$$(s^2 + s - 2)\mathcal{L}\{y\} = \frac{6(s^2+9) + 3s-33}{s^2+9} = \frac{6s^2+3s+21}{s^2+9}$$

$$\mathcal{L}\{y\} = \frac{6s^2+3s+21}{(s^2+9)(s^2+s-2)}$$

and

$$y = \mathcal{L}^{-1} \left\{ \frac{6s^2 + 3s + 21}{(s^2 + 9)(s-1)(s+2)} \right\}$$

Let
$$\frac{6s^2 + 3s + 21}{(s^2 + 9)(s-1)(s+2)} \equiv \frac{A}{(s+2)} + \frac{B}{(s-1)} + \frac{Cs+D}{(s^2 + 9)}$$

$$= \frac{A(s-1)(s^2 + 9) + B(s+2)(s^2 + 9) + (Cs+D)(s+2)(s-1)}{(s+2)(s-1)(s^2 + 9)}$$

Hence,
$$6s^2 + 3s + 21 = A(s-1)(s^2 + 9) + B(s+2)(s^2 + 9) + (Cs+D)(s+2)(s-1)$$

When $s = 1$: $6 + 3 + 21 = B(3)(10)$ i.e. $30 = 30B$ and $B = 1$

When $s = -2$: $24 - 6 + 21 = A(-3)(13)$ i.e. $39 = -39A$ and $A = -1$

Equating s^3 coefficients: $0 = A + B + C$ i.e. $C = 0$

Equating constant terms: $21 = -9A + 18B - 2D$ i.e. $21 = 9 + 18 - 2D$ and $D = 3$

Hence,
$$y = \mathcal{L}^{-1} \left\{ \frac{-1}{(s+2)} + \frac{1}{(s-1)} + \frac{3}{(s^2 + 3^2)} \right\}$$

i.e.
$$y = -e^{-2x} + e^x + \sin 3x \quad \text{or} \quad y = e^x - e^{-2x} + \sin 3x$$

9. Use Laplace transforms to solve the differential equation:

$$\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y = 3e^x \cos 2x, \text{ given } y(0) = 2 \text{ and } y'(0) = 5$$

$$\mathcal{L} \left\{ \frac{d^2 y}{dx^2} \right\} - 2\mathcal{L} \left\{ \frac{dy}{dx} \right\} + 2\mathcal{L}\{y\} = \mathcal{L}\{3e^x \cos 2x\}$$

i.e.
$$[s^2 \mathcal{L}\{y\} - s y(0) - y'(0)] - 2[s \mathcal{L}\{y\} - y(0)] + 2\mathcal{L}\{y\} = 3 \left(\frac{s-1}{(s-1)^2 + 2^2} \right)$$

$y(0) = 2$ and $y'(0) = 5$, hence

$$s^2 \mathcal{L}\{y\} - 2s - 5 - 2s \mathcal{L}\{y\} + 4 + 2\mathcal{L}\{y\} = 3 \left(\frac{s-1}{(s-1)^2 + 2^2} \right)$$

i.e.
$$(s^2 - 2s + 2)\mathcal{L}\{y\} = 2s + 1 + 3 \left(\frac{s-1}{(s-1)^2 + 2^2} \right)$$

$$\begin{aligned}
&= \frac{(3s-3)+(2s+1)(s^2-2s+5)}{(s^2-2s+5)} \\
&= \frac{3s-3+2s^3-4s^2+10s+s^2-2s+5}{(s^2-2s+5)} \\
&= \frac{2s^3-3s^2+11s+2}{(s^2-2s+5)}
\end{aligned}$$

from which,

$$\mathcal{L}\{y\} = \frac{2s^3-3s^2+11s+2}{(s^2-2s+5)(s^2-2s+2)}$$

and

$$y = \mathcal{L}^{-1}\left\{\frac{2s^3-3s^2+11s+2}{(s^2-2s+5)(s^2-2s+2)}\right\}$$

$$\begin{aligned}
\text{Let } \frac{2s^3-3s^2+11s+2}{(s^2-2s+5)(s^2-2s+2)} &\equiv \frac{As+B}{(s^2-2s+5)} + \frac{Cs+D}{(s^2-2s+2)} \\
&= \frac{(As+B)(s^2-2s+2) + (Cs+D)(s^2-2s+5)}{(s^2-2s+5)(s^2-2s+2)}
\end{aligned}$$

$$\text{Hence, } 2s^3-3s^2+11s+2 = (As+B)(s^2-2s+2) + (Cs+D)(s^2-2s+5)$$

$$\text{Equating } s^3 \text{ coefficients: } 2 = A + C \quad (1)$$

$$\text{Equating } s^2 \text{ coefficients: } -3 = -2A + B - 2C + D \quad (2)$$

$$\text{Equating } s \text{ coefficients: } 11 = 2A - 2B + 5C - 2D \quad (3)$$

$$\text{Equating constant terms: } 2 = 2B + 5D \quad (4)$$

From (1), $C = 2 - A$ and

$$\text{substituting } C = 2 - A \text{ in (2): } -3 = -2A + B - 2(2 - A) + D$$

$$\text{i.e. } 1 = B + D \quad (5)$$

$$2 \times (5) \text{ gives: } 2 = 2B + 2D \quad (6)$$

$$(4) - (6) \text{ gives: } 0 = 3D \quad \text{i.e. } \mathbf{D = 0}$$

From (4), if $D = 0$, then $\mathbf{B = 1}$

$$\text{In (2), if } B = 1 \text{ and } D = 0, \text{ then } -3 = -2A + 1 - 2C$$

$$\text{i.e. } 4 = 2A + 2C \quad (7)$$

$$\text{From (3), } 11 = 2A - 2 + 5C$$

$$\text{i.e.} \quad 13 = 2A + 5C \quad (8)$$

$$(8) - (7) \text{ gives:} \quad 9 = 3C \quad \text{i.e.} \quad C = 3$$

In (1), if $C = 3$, then $A = -1$

$$\begin{aligned} \text{Hence,} \quad y &= \mathcal{L}^{-1} \left\{ \frac{1-s}{(s^2-2s+5)} + \frac{3s}{(s^2-2s+2)} \right\} \\ &= -\mathcal{L}^{-1} \left\{ \frac{s-1}{(s-1)^2+2^2} \right\} + \mathcal{L}^{-1} \left\{ \frac{3s}{(s-1)^2+1^2} \right\} \\ &= -\mathcal{L}^{-1} \left\{ \frac{s-1}{(s-1)^2+2^2} \right\} + \mathcal{L}^{-1} \left\{ \frac{3(s-1)+3}{(s-1)^2+1^2} \right\} \\ &= -\mathcal{L}^{-1} \left\{ \frac{s-1}{(s-1)^2+2^2} \right\} + \mathcal{L}^{-1} \left\{ \frac{3(s-1)}{(s-1)^2+1^2} \right\} + \mathcal{L}^{-1} \left\{ \frac{3}{(s-1)^2+1^2} \right\} \end{aligned}$$

$$\text{i.e.} \quad y = -e^x \cos 2x + 3e^x \cos x + 3e^x \sin x \quad \text{or} \quad y = 3e^x (\cos x + \sin x) - e^x \cos 2x$$

10. The free oscillations of a lightly damped elastic system are given by the equation:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0$$

where y is the displacement from the equilibrium position. If when time $t = 0$, $y = 2$ and $\frac{dy}{dt} = 0$, determine an expression for the displacement.

Taking Laplace transforms of each term of $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0$

$$\text{gives:} \quad \mathcal{L} \left\{ \frac{d^2y}{dt^2} \right\} + 2\mathcal{L} \left\{ \frac{dy}{dt} \right\} + 5\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$\text{Hence,} \quad [s^2 \mathcal{L}\{y\} - sy(0) - y'(0)] + 2[s\mathcal{L}\{y\} - y(0)] + 5\mathcal{L}\{y\} = 0$$

$$\text{Since} \quad y(0) = 2 \quad \text{and} \quad y'(0) = 8$$

$$\text{then} \quad [s^2 \mathcal{L}\{y\} - 2s] + 2[s\mathcal{L}\{y\} - 2] + 5\mathcal{L}\{y\} = 0$$

$$s^2 \mathcal{L}\{y\} - 2s + 2s\mathcal{L}\{y\} - 4 + 5\mathcal{L}\{y\} = 0$$

$$\text{i.e.} \quad (s^2 + 2s + 5) \mathcal{L}\{y\} = 2s + 4$$

$$\mathcal{L}\{y\} = \frac{2s+4}{(s^2+2s+5)}$$

i.e.

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{\frac{2s+4}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{2s+4}{(s+1)^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s+1)+2}{(s+1)^2+4}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{2(s+1)}{(s+1)^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\} \\ &= 2\mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\} \end{aligned}$$

i.e. $y = 2e^{-t} \cos 2t + e^{-t} \sin 2t$ or $y = e^{-t}(2 \cos 2t + \sin 2t)$

11. Solve, using Laplace transforms, Problems 4 to 9 of Exercise 203, page 556 and Problems 1 to 6 of Exercise 204, page 558.

These Problems are left as an exercise for the learner.

12. Solve, using Laplace transforms, Problems 3 to 6 of Exercise 205, page 563, Problems 5 and 6 of Exercise 206, page 565, Problems 4 and 7 of Exercise 207, page 567, and Problems 5 and 6 of Exercise 208, page 569.

These Problems are left as an exercise for the learner.

CHAPTER 72 THE SOLUTION OF SIMULTANEOUS DIFFERENTIAL EQUATIONS USING LAPLACE TRANSFORM

EXERCISE 258 Page 761

1. Solve the following pair of simultaneous differential equations:

$$2 \frac{dx}{dt} + \frac{dy}{dt} = 5e^t$$

$$\frac{dy}{dt} - 3 \frac{dx}{dt} = 5 \quad \text{given that when } t = 0, x = 0 \text{ and } y = 0$$

Taking Laplace transforms of each term in each equation gives:

$$2[s\mathcal{L}\{x\} - y(0)] + [s\mathcal{L}\{y\} - y(0)] = \frac{5}{s-1}$$

$$[s\mathcal{L}\{y\} - y(0)] - 3[s\mathcal{L}\{x\} - x(0)] = \frac{5}{s}$$

$x(0) = 0$ and $y(0) = 0$, hence

$$2s\mathcal{L}\{x\} + s\mathcal{L}\{y\} = \frac{5}{s-1} \quad (1)$$

$$\text{and} \quad -3s\mathcal{L}\{x\} + s\mathcal{L}\{y\} = \frac{5}{s} \quad (2)$$

$$(1) - (2) \text{ gives: } 5s\mathcal{L}\{x\} = \frac{5}{(s-1)} - \frac{5}{s} = \frac{5s - 5(s-1)}{s(s-1)} = \frac{5}{s(s-1)}$$

$$\text{i.e.} \quad \mathcal{L}\{x\} = \frac{5}{5s^2(s-1)} = \frac{1}{s^2(s-1)}$$

$$\text{and} \quad x = \mathcal{L}^{-1} \left\{ \frac{1}{s^2(s-1)} \right\}$$

$$\text{Let } \frac{1}{s^2(s-1)} \equiv \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} = \frac{A(s)(s-1) + B(s-1) + Cs^2}{s^2(s-1)}$$

$$\text{from which,} \quad 1 = A(s)(s-1) + B(s-1) + Cs^2$$

$$\text{When } s = 0: \quad 1 = -B \quad \text{i.e.} \quad B = -1$$

$$\text{When } s = 1: \quad 1 = C \quad \text{i.e.} \quad C = 1$$

Equating s^2 coefficients: $0 = A + C$ i.e. $A = -1$ since $C = 1$

Hence,
$$x = \mathcal{L}^{-1} \left\{ -\frac{1}{s} - \frac{1}{s^2} + \frac{1}{(s-1)} \right\}$$

i.e.
$$x = -1 - t + e^t \quad \text{or} \quad \mathbf{x} = \mathbf{e}^t - \mathbf{t} - \mathbf{1}$$

From equation (1)
$$2s\mathcal{L}\{x\} + s\mathcal{L}\{y\} = \frac{5}{s-1}$$

i.e.
$$2s\mathcal{L}\{e^t - t - 1\} + s\mathcal{L}\{y\} = \frac{5}{s-1}$$

i.e.
$$2s \left(\frac{1}{s-1} - \frac{1}{s^2} - \frac{1}{s} \right) + s\mathcal{L}\{y\} = \frac{5}{s-1}$$

i.e.
$$\frac{2s}{(s-1)} - \frac{2}{s} - 2 + s\mathcal{L}\{y\} = \frac{5}{s-1}$$

i.e.
$$s\mathcal{L}\{y\} = \frac{5}{(s-1)} - \frac{2s}{(s-1)} + \frac{2}{s} + 2$$

and
$$s\mathcal{L}\{y\} = \frac{5-2s}{(s-1)} + \frac{2}{s} + 2$$

i.e.
$$\mathcal{L}\{y\} = \frac{5-2s}{s(s-1)} + \frac{2}{s^2} + \frac{2}{s}$$

and
$$y = \mathcal{L}^{-1} \left\{ \frac{5-2s}{s(s-1)} + \frac{2}{s^2} + \frac{2}{s} \right\}$$

Let
$$\frac{5-2s}{s(s-1)} = \frac{A}{s} + \frac{B}{(s-1)} = \frac{A(s-1) + Bs}{s(s-1)}$$

from which,
$$5 - 2s = A(s-1) + Bs$$

When $s = 0$:
$$5 = -A \quad \text{i.e.} \quad A = -5$$

When $s = 1$:
$$3 = B$$

Hence,
$$y = \mathcal{L}^{-1} \left\{ -\frac{5}{s} + \frac{3}{(s-1)} + \frac{2}{s^2} + \frac{2}{s} \right\}$$

i.e.
$$y = \mathcal{L}^{-1} \left\{ \frac{3}{(s-1)} + \frac{2}{s^2} - \frac{3}{s} \right\}$$

i.e.
$$\mathbf{y} = \mathbf{3e}^t + \mathbf{2t} - \mathbf{3}$$

2. Solve the following pair of simultaneous differential equations:

$$2 \frac{dy}{dt} - y + x + \frac{dx}{dt} - 5 \sin t = 0$$

$$3 \frac{dy}{dt} + x - y + 2 \frac{dx}{dt} - e^t = 0 \text{ given that at } t=0, x=0 \text{ and } y=0$$

Taking Laplace transforms of each term in each equation gives:

$$2[s\mathcal{L}\{y\} - y(0)] - \mathcal{L}\{y\} + \mathcal{L}\{x\} + [s\mathcal{L}\{x\} - x(0)] - \frac{5}{s^2+1} = 0$$

$$3[s\mathcal{L}\{y\} - y(0)] + \mathcal{L}\{x\} - \mathcal{L}\{y\} + 2[s\mathcal{L}\{x\} - x(0)] - \frac{1}{s-1} = 0$$

$y(0) = 0$ and $x(0) = 0$, hence

$$(2s-1)\mathcal{L}\{y\} + (s+1)\mathcal{L}\{x\} = \frac{5}{s^2+1} \quad (1)$$

and $(3s-1)\mathcal{L}\{y\} + (2s+1)\mathcal{L}\{x\} = \frac{1}{s-1} \quad (2)$

$$(3s-1) \times (1) \text{ gives: } (3s-1)(2s-1)\mathcal{L}\{y\} + (3s-1)(s+1)\mathcal{L}\{x\} = (3s-1)\frac{5}{s^2+1} \quad (3)$$

$$(2s-1) \times (2) \text{ gives: } (2s-1)(3s-1)\mathcal{L}\{y\} + (2s-1)(2s+1)\mathcal{L}\{x\} = (2s-1)\frac{1}{s-1} \quad (4)$$

$$(3) - (4) \text{ gives: } \left[(3s^2+2s-1) - (4s^2-1) \right] \mathcal{L}\{x\} = \frac{5(3s-1)}{s^2+1} - \frac{2s-1}{s-1} \quad (5)$$

$$\text{i.e. } (-s^2+2s)\mathcal{L}\{x\} = \frac{5(3s-1)(s-1) - (2s-1)(s^2+1)}{(s-1)(s^2+1)}$$

$$\begin{aligned} \text{i.e. } \mathcal{L}\{x\} &= - \left\{ \frac{15s^2 - 20s + 5 - 2s^3 - 2s + s^2 + 1}{s(s-2)(s-1)(s^2+1)} \right\} \\ &= \left\{ \frac{2s^3 - 16s^2 + 22s - 6}{s(s-2)(s-1)(s^2+1)} \right\} \end{aligned}$$

and $x = \mathcal{L}^{-1} \left\{ \frac{2s^3 - 16s^2 + 22s - 6}{s(s-2)(s-1)(s^2+1)} \right\}$

$$\text{Let } \frac{2s^3 - 16s^2 + 22s - 6}{s(s-2)(s-1)(s^2+1)} \equiv \frac{A}{s} + \frac{B}{(s-2)} + \frac{C}{(s-1)} + \frac{Ds+E}{(s^2+1)}$$

$$= \frac{A(s-2)(s-1)(s^2+1) + B(s)(s-1)(s^2+1) + C(s)(s-2)(s^2+1) + (Ds+E)(s)(s-2)(s-1)}{s(s-2)(s-1)(s^2+1)}$$

from which, $2s^3 - 16s^2 + 22s - 6 = A(s-2)(s-1)(s^2+1) + B(s)(s-1)(s^2+1) + C(s)(s-2)(s^2+1) + (Ds+E)(s)(s-2)(s-1)$

When $s = 0$: $-6 = A(-2)(-1)(1)$ i.e. $A = -3$

When $s = 1$: $2 - 16 + 22 - 6 = C(1)(-1)(2)$ i.e. $C = -1$

When $s = 2$: $16 - 64 + 44 - 6 = B(2)(1)(5)$ i.e. $B = -1$

Equating s^4 coefficients: $0 = A + B + C + D$ i.e. $D = 5$

Equating s^3 coefficients: $2 = -3A - B - 2C - 3D + E$

i.e. $2 = 9 + 1 + 2 - 15 + E$ i.e. $E = 5$

Hence,
$$x = \mathcal{L}^{-1} \left\{ -\frac{3}{s} - \frac{1}{(s-2)} - \frac{1}{(s-1)} + \frac{5s+5}{(s^2+1)} \right\}$$

i.e. $x = -3 - e^{-2t} - e^t + 5 \cos t + 5 \sin t$ or $x = 5 \cos t + 5 \sin t - e^{2t} - e^t - 3$

From equations (1) and (2)

$(2s+1) \times (1)$ gives: $(2s+1)(2s-1)\mathcal{L}\{y\} + (2s+1)(s+1)\mathcal{L}\{x\} = (2s+1)\frac{5}{s^2+1}$ (6)

$(s+1) \times (2)$ gives: $(s+1)(3s-1)\mathcal{L}\{y\} + (s+1)(2s+1)\mathcal{L}\{x\} = (s+1)\frac{1}{s-1}$ (7)

$(6) - (7)$: $\left[(4s^2-1) - (3s^2+2s-1) \right] \mathcal{L}\{y\} = \frac{5(2s+1)}{s^2+1} - \frac{(s+1)}{(s-1)} = \frac{(10s+5)(s-1) - (s+1)(s^2+1)}{(s-1)(s^2+1)}$

i.e. $(s^2-2s)\mathcal{L}\{y\} = \frac{10s^2-5s-5-s^3-s-s^2-1}{(s-1)(s^2+1)}$

and
$$\mathcal{L}\{y\} = \frac{-s^3+9s^2-6s-6}{s(s-2)(s-1)(s^2+1)}$$

and
$$y = \mathcal{L}^{-1} \left\{ \frac{-s^3+9s^2-6s-6}{s(s-1)(s-2)(s^2+1)} \right\}$$

Let $\frac{-s^3+9s^2-6s-6}{s(s-1)(s-2)(s^2+1)} \equiv \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s-2)} + \frac{Ds+E}{(s^2+1)}$

$$= \frac{A(s-1)(s-2)(s^2+1) + B(s)(s-2)(s^2+1) + C(s)(s-1)(s^2+1) + (Ds+E)(s)(s-1)(s-2)}{s(s-1)(s-2)(s^2+1)}$$

from which, $-s^3 + 9s^2 - 6s - 6 = A(s-1)(s-2)(s^2+1) + B(s)(s-2)(s^2+1) + C(s)(s-1)(s^2+1) + (Ds+E)(s)(s-1)(s-2)$

When $s = 0$: $-6 = A(-1)(-2)(1)$ i.e. $A = -3$

When $s = 1$: $-1 + 9 - 6 - 6 = B(1)(-1)(2)$ i.e. $B = 2$

When $s = 2$: $-8 + 36 - 12 - 6 = C(2)(1)(5)$ i.e. $C = 1$

Equating s^4 coefficients: $0 = A + B + C + D$ i.e. $D = 0$

Equating s^3 coefficients: $-1 = -3A - 2B - C - 3D + E$

i.e. $-1 = 9 - 4 - 1 + 0 + E$ i.e. $E = -5$

Hence, $y = \mathcal{L}^{-1} \left\{ -\frac{3}{s} + \frac{2}{(s-1)} + \frac{1}{(s-2)} - \frac{5}{(s^2+1)} \right\}$

i.e. $y = -3 + 2e^t + e^{2t} - 5 \sin t$ or $y = e^{2t} + 2e^t - 3 - 5 \sin t$

3. Solve the following pair of simultaneous differential equations:

$$\frac{d^2x}{dt^2} + 2x = y$$

$$\frac{d^2y}{dt^2} + 2y = x \quad \text{given that at } t = 0, x = 4, y = 2, \frac{dx}{dt} = 0 \text{ and } \frac{dy}{dt} = 0$$

Taking Laplace transforms of each term in each equation gives:

$$[s^2 \mathcal{L}\{x\} - s x(0) - x'(0)] + 2\mathcal{L}\{x\} = \mathcal{L}\{y\}$$

and $[s^2 \mathcal{L}\{y\} - s y(0) - y'(0)] + 2\mathcal{L}\{y\} = \mathcal{L}\{x\}$

$x(0) = 4$ and $x'(0) = 0$, hence $[s^2 \mathcal{L}\{x\} - 4s] + 2\mathcal{L}\{x\} = \mathcal{L}\{y\}$

$y(0) = 2$ and $y'(0) = 0$, hence $[s^2 \mathcal{L}\{y\} - 2s] + 2\mathcal{L}\{y\} = \mathcal{L}\{x\}$

i.e. $(s^2 + 2)\mathcal{L}\{x\} - \mathcal{L}\{y\} = 4s$ (1)

and $-\mathcal{L}\{x\} + (s^2 + 2)\mathcal{L}\{y\} = 2s$ (2)

$$(s^2+2) \times (2) \text{ gives: } -(s^2+2)\mathcal{L}\{x\} + (s^2+2)(s^2+2)\mathcal{L}\{y\} = 2s(s^2+2) \quad (3)$$

$$(1) + (3) \text{ gives: } \left[(s^2+2)^2 - 1 \right] \mathcal{L}\{y\} = 2s(s^2+2) + 4s$$

$$\text{i.e. } (s^4 + 4s^2 + 3) \mathcal{L}\{y\} = 2s^3 + 8s$$

$$\text{and } \mathcal{L}\{y\} = \frac{2s^3 + 8s}{(s^4 + 4s^2 + 3)} = \frac{2s^3 + 8s}{(s^2 + 3)(s^2 + 1)}$$

$$\text{and } y = \mathcal{L}^{-1} \left\{ \frac{2s^3 + 8s}{(s^2 + 3)(s^2 + 1)} \right\}$$

$$\text{Let } \frac{2s^3 + 8s}{(s^2 + 3)(s^2 + 1)} = \frac{As + B}{(s^2 + 3)} + \frac{Cs + D}{(s^2 + 1)} = \frac{(As + B)(s^2 + 1) + (Cs + D)(s^2 + 3)}{(s^2 + 3)(s^2 + 1)}$$

$$\text{from which, } 2s^3 + 8s = (As + B)(s^2 + 1) + (Cs + D)(s^2 + 3)$$

$$\text{Equating } s^3 \text{ coefficients: } 2 = A + C \quad (4)$$

$$\text{Equating } s^2 \text{ coefficients: } 0 = B + D \quad (5)$$

$$\text{Equating } s \text{ coefficients: } 8 = A + 3C \quad (6)$$

$$(6) - (4) \text{ gives: } 6 = 2C \quad \text{i.e. } C = 3$$

$$\text{and from (4), } A = 2 - C \quad \text{i.e. } A = -1$$

$$\text{Equating constant terms: } 0 = B + 3D \quad (7)$$

$$(7) - (5) \text{ gives: } 0 = 2D \quad \text{i.e. } D = 0 \quad \text{and from (5), } B = 0$$

$$\text{Hence, } y = \mathcal{L}^{-1} \left\{ \frac{-s}{(s^2 + 3)} + \frac{3s}{(s^2 + 1)} \right\} = \mathcal{L}^{-1} \left\{ -\frac{s}{(s^2 + (\sqrt{3})^2)} + \frac{3s}{(s^2 + (\sqrt{1})^2)} \right\}$$

$$\text{i.e. } y = -\cos(\sqrt{3}t) + 3\cos t \quad \text{or } y = 3\cos t - \cos(\sqrt{3}t)$$

$$\text{If } y = 3\cos t - \cos(\sqrt{3}t) \quad \text{then } \frac{dy}{dt} = -3\sin t + \sqrt{3}\sin\sqrt{3}t$$

$$\text{and } \frac{d^2y}{dt^2} = -3\cos t + 3\cos\sqrt{3}t$$

Since from one of the original equations,

$$\frac{d^2y}{dt^2} + 2y = x$$

then $-3 \cos t + 3 \cos(\sqrt{3} t) + 2(3 \cos t - \cos(\sqrt{3} t)) = x$

i.e. $x = -3 \cos t + 3 \cos(\sqrt{3} t) + 6 \cos t - 2 \cos(\sqrt{3} t)$

i.e. $x = 3 \cos t + \cos(\sqrt{3} t)$
